1. Increasing and Decreasing Functions

**Definition**  A function $f$ is (strictly) **increasing** on an interval $I$ if for every $x_1, x_2$ in $I$ with $x_1 < x_2$, 
$$f(x_1) < f(x_2).$$

A function $f$ is (strictly) **decreasing** on an interval $I$ if for every $x_1, x_2$ in $I$ with $x_1 < x_2$, 
$$f(x_2) < f(x_1).$$

**Example**  The graph of $f$ is given below. Determine graphically the interval on which $f$ is increasing.

![Graph of f](image)

$f(x)$ is increasing on $(0, 0.8), (1.5, 4)$

How can we determine algebraically where $f$ is increasing and where $f$ is decreasing? **Observe that** the graph of $f$ has positive slope on the intervals: $(0, 0.8), (1.5, 4)$.

**Theorem**  Suppose that $f$ is differentiable on an interval $I$.

a. If $f'(x) > 0$ for all $x$ in $I$, then $f$ is increasing on $I$.

b. If $f'(x) < 0$ for all $x$ in $I$, then $f$ is decreasing on $I$.

The proof is directly from the Mean Value Theorem. Let $x_1$ and $x_2$ be in $I$ and $x_1 < x_2$. Then by the Mean Value Theorem, we know there exists a value $c$ in $(x_1, x_2)$ such that 
$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$ 

If $f'(c) > 0$, then $\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$ that implies $f(x_2) - f(x_1) > 0$ because $x_2 > x_1$. Hence, 
$$f(x_2) > f(x_1)$$ 

$f$ is increasing. In a similar way, we can show b.

**Example**  The graph of $f'$ is given below. Determine graphically the interval on which $f$ is increasing.
Example  Find the intervals where \( f(x) = 2x^3 + 9x^2 - 24x - 10 \) is increasing and decreasing. Verify your answers by graphing both \( f \) and \( f' \).

Compute \( f' \):

\[
f'(x) = 6x^2 + 18x - 24 = 6(x^2 + 3x - 4) = 6(x + 4)(x - 1)
\]

Check signs of \( f' \):

Know \( f'(x) = 0 \) when \( x = -4 \) and \( x = 1 \).

<table>
<thead>
<tr>
<th>interval</th>
<th>(-\infty, -4)</th>
<th>(-4, 1)</th>
<th>(1, \infty)</th>
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<td>( f'(x) )</td>
<td>+</td>
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So, \( f \) is increasing on \( (-\infty, -4) \), \( (1, \infty) \) and is decreasing on \( (-4, 1) \).
Example  Find the intervals where \( f(x) = \ln(x^2 - 4) \) is increasing and decreasing. Verify your answers by graphing both \( f \) and \( f' \).

Domain of \( f \): \( x^2 - 4 > 0 \), \( x^2 > 4 \), \( |x| > \sqrt{4} = 2 \), \( x > 2 \) or \( x < -2 \).

Compute \( f' \):

\[
 f'(x) = \frac{1}{x^2 - 4}(2x) = \frac{2x}{(x-2)(x+2)}
\]

Check signs of \( f' \): Know \( f'(x) = 0 \) when \( x = 0 \), and \( f' \) is not defined when \( x = 2 \) and \( x = -2 \).

Check signs of \( f' \) over intervals: \( (-\infty, -2) \), \( (2, \infty) \)

\[
\begin{align*}
 f'(-3) &= \frac{-6}{(-5)(-1)} < 0, \\
 f'(3) &= \frac{6}{(1)(5)} > 0
\end{align*}
\]

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So, \( f \) is increasing on \( (2, \infty) \) and is decreasing on \( (-\infty, -2) \).

2. First Derivative Test For Local Extrema:

**Theorem** Suppose that \( f \) is continuous on \( [a, b] \) and \( c \) in \( (a, b) \) is a critical number.

a. If \( f' \) changes sign from positive to negative at \( x = c \), then \( f(c) \) is a **local maximum** of \( f \). The number \( c \) is called a **local maximum point**.

b. If \( f' \) changes sign from negative to positive at \( x = c \), then \( f(c) \) is a **local minimum** of \( f \). The number \( c \) is called a **local minimum point**.

c. If \( f' \) does not change sign at \( x = c \), then \( f(c) \) is not a local extremum.

**Steps** for locating the \( x \)-coordinate of a local minimum point or a local maximum point:

a. Find **all critical numbers** \( c \) of \( f \) and values \( x \) which are **not in the domain** of \( f \) but at which \( f' \) is not defined.
b. Determine the sign changes of $f'$ over the intervals whose end points are critical numbers $c$ and $\bar{x}$ (except $\pm \infty$).

c. If $f'$ changes from positive to negative at $c$, then $x = c$ is a local maximum point. If $f'$ changes from negative to positive at $c$, then $x = c$ is a local maximum point.

**Example** Find the $x$–coordinates of all extrema of $f(x) = x^2 e^{-3x}$.

Compute $f'$ and find all critical numbers:

$$f'(x) = 2xe^{-3x} - 3x^2e^{-3x} = xe^{-3x}(2 - 3x), \quad f'(x) = 0 \iff \begin{cases} x = 0 \\ 2 - 3x = 0, \quad x = \frac{2}{3} \end{cases}$$

$f'$ is defined everywhere. So, there is no $c$ at which $f'$ is not defined.

Check the sign change of $f'$ over intervals: $\left(-\infty, 0\right), \left(0, \frac{2}{3}\right), \left(\frac{2}{3}, \infty\right)$

$$\begin{array}{c|ccc}
\text{interval} & \left(-\infty, 0\right) & \left(0, \frac{2}{3}\right) & \left(\frac{2}{3}, \infty\right) \\
\text{sign of } f' & - & + & - \\
\end{array}$$

$f'$ has a local maximum value at $x = \frac{2}{3}$ and a local minimum value at $x = 0$.

**Example** Find the $x$–coordinates of all extrema of $f(x) = \frac{x^2}{x^2 - 4}$.

Compute $f'$ and find all critical numbers:

$$f'(x) = \frac{2x(x^2 - 4) - x^2(2x)}{(x^2 - 4)^2} = \frac{2x(x^2 - 4 - x^2)}{(x^2 - 4)^2} = \frac{-8x}{(x^2 - 4)^2} = 0 \iff x = 0$$

$f'$ is not defined at $x^2 - 4 = 0$, $x = \pm 2$. However, since $(x^2 - 4)^2 > 0$ for all $x \neq \pm 2$, no sign change for the denominator of $f'$. So, we only need to check the sign change of the numerator of $f'$.

Check sign changes of $f'$ over $\left(-\infty, 0\right), \left(0, \infty\right)$:

$f'(x) > 0$ for $x$ in $\left(-\infty, 0\right)$ and $f'(x) < 0$ for $x$ in $\left(0, \infty\right)$.

$x = 0$ is a local maximum point.
Example  Find the x-coordinates of all extrema of \( f(x) = \sin x - x \).

Compute \( f' \) and find all critical numbers:

\[
f'(x) = \cos x - 1 = 0, \quad \text{cos} x = 1, \quad x = 0, \quad x = 2n\pi \text{ and } x = -2n\pi
\]

Critical numbers:

\[
x = 0, \quad x = \pm 2\pi, \quad x = \pm 4\pi, \ldots
\]

Check signs of \( f' \): since \(-1 \leq \cos x \leq 1, \quad -2 \leq \cos x - 1 \leq 0\). So, \( f'(x) \leq 0 \) for all \( x \) in \((-\infty, \infty)\), that is \( f' \) does not change sign at its critical numbers and these critical numbers are not local extrema.

Example  Find the local extrema of \( f(x) = x^{5/3} - 3x^{2/3} \).

Domain of \( f \) : \( D_f = (-\infty, \infty) \)

Compute \( f' \) and find all critical numbers of \( f \):

\[
f'(x) = \frac{5}{3} x^{2/3} - 2x^{-1/3} = x^{-1/3}(\frac{5}{3} x - 2)
\]

\[
f'(x) = 0 \quad \text{if}
\]
\[ \frac{5}{3}x - 2 = 0, \quad x = \frac{6}{5} \]

\( f'(x) \) is not defined if \( x = 0 \).

Check signs of \( f'(x) \) on the intervals: \( (-\infty, 0) \), \( (0, \frac{6}{5}) \), \( \left( \frac{6}{5}, \infty \right) \)

\[
f'(-1) = \frac{1}{\sqrt{(-1)}} \left( \frac{5}{3}(-1) - 2 \right) = \frac{11}{3}, \quad f'(1) = \frac{1}{\sqrt{1}} \left( \frac{5}{3} - 2 \right) = -\frac{1}{3}
\]

\[
f'(2) = \frac{1}{\sqrt{2}} \left( \frac{5}{3}(2) - 2 \right) = \frac{2}{3} 2^{\frac{3}{2}}
\]

<table>
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So, \( f(0) \) is a local maximum, \( f\left( \frac{6}{5} \right) \) is a local minimum.

**Example** Sketch a graph of a function with the given properties:

\( f(0) = 1, \quad f(2) = 5 \)

\( f'(0) = 0, \quad f'(2) = 0 \)

\( f'(x) < 0, \quad \text{for } x < 0 \text{ and } x > 2; \quad f'(x) > 0 \text{ for } 0 < x < 2 \)
$f(3) = 0, \ f(0) \ does \ not \ exist$

$f'(3) = 0, \ f'(0) \ does \ not \ exist$

$f'(x) < 0, \ for \ x < 0 \ and \ x > 3; \ f'(x) > 0 \ for \ 0 < x < 3$