1.2 The Concepts of Limit

What is a limit (of a sequence of real numbers)?

Example: a sequence of numbers:

\[
\begin{align*}
\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{999}{1000}, \ldots, \frac{9999}{10000}, \ldots, \frac{99999}{100000}, \ldots
\end{align*}
\]

- Is there an end of this sequence? No.
- Can you think of a number that these terms are getting closer and closer to? Yes, the number is 1. That is,

\[
\lim_{n \to \infty} \frac{n}{n + 1} = 1 \quad (1 \text{ is not in } \left\{ \frac{n}{n + 1} \right\})
\]

or these numbers are approaching to 1 as \( n \) is getting larger and larger.

Example: a sequence of numbers:

\[
0, 1, 0, 1, 0, 1, 0, 1, \ldots
\]

- Is there an end of this sequence? No.
- Can you think of a number that these terms are getting closer and closer to? No.

The limit \( L \) of a sequence \( \{x_n\} \) is a real number approached by \( x_n \).
Example: Consider \( f(x) = \frac{x^2 - 5}{x - 2} \) and \( g(x) = \frac{x^2 - 4}{x - 2} \).

Both functions are not defined at \( x = 2 \). What can we say about \( f(x) \) and \( g(x) \) as \( x \) is getting closer and closer to 2?

- As \( x \) is getting closer to 2:
  \[
x = 1.99, 1.999, 1.9999, 1.99999, 1.999999, \ldots
  \]
  values of \( f(x) \) (y values) are:
  103.99, 1003.999, 1.0003.9999, 1000 03.99999, 1000 003.999999

  \( \lim_{x \to 2^-} f(x) = \infty \)

- As \( x \) is getting closer to 2:
  \[
x = 2.01, 2.001, 2.0001, 2.00001, 2.000001, \ldots
  \]
  values of \( f(x) \) (y values) are:
  - 95.99, -995.999, -9995.9999, -99995.99999, \ldots

  \( \lim_{x \to 2^+} f(x) = -\infty \)

But \( \lim_{x \to 2} f(x) \) does not exist.
\[ g(x) = \frac{x^2 - 4}{x - 2} \]

- As \( x \) is getting closer to 2:
  \[ x = 1.99, 1.999, 1.9999, 1.99999, 1.999999, \ldots \]
  values of \( g(x) \) (y values) are:
  \[ 3.99, 3.999, 3.9999, 3.99999, 3.999999, \ldots \]
  \[ \lim_{x \to 2^-} g(x) = 4 \]

- As \( x \) is getting closer to 2:
  \[ x = 2.01, 2.001, 2.0001, 2.00001, 2.000001, \ldots \]
  values of \( g(x) \) (y values) are:
  \[ 4.01, 4.001, 4.0001, 4.00001, 4.000001, \ldots \]
  \[ \lim_{x \to 2^+} g(x) = 4 \]

So, \( \lim_{x \to 2} g(x) = 4 \).
Notations:

i. \( x \to a^- \) means \( x \) approaches \( a \) from the left side of \( a \).

ii. \( x \to a^+ \) means \( x \) approaches \( a \) from the right side of \( a \).

iii. \( \lim_{x \to a^-} f(x) \) denotes the limit of \( f(x) \) as \( x \) approaches \( a \) from the left side.

iv. \( \lim_{x \to a^+} f(x) \) denotes the limit of \( f(x) \) as \( x \) approaches \( a \) from the right side.

v. \( \lim_{x \to a} f(x) \) denotes the limit of \( f(x) \) as \( x \) approaches \( a \) from both left and right sides.

Let \( L \) be a finite number.

Definition: \( \lim_{x \to a} f(x) = L \) if and only if

\[
\lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L.
\]
Compute Limits Graphically:

Example: The graph of \( f(x) \) is given below. Find limits.

\[ a. \lim_{x \to 1^-} f(x) = \]
\[ b. \lim_{x \to 1^+} f(x) = \]
\[ c. \lim_{x \to 1} f(x) = \]
\[ d. \lim_{x \to 2^-} f(x) = \]
\[ e. \lim_{x \to 2^+} f(x) = \]
\[ f. \lim_{x \to 2} f(x) = \]

Example: (http://curvebank.calstatela.edu/limit/limit.htm)
Compute Limits Numerically:

**Example:** Evaluate \( \lim_{x \to 0} \frac{\sin x}{x} \) numerically and graphically.

Numerically:

Note that \( x \) is in radians. (TI-83, TI-89: (1) change mode to radians; (2) set table/independent variable to ask instead of auto.)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \frac{\sin x}{x} )</th>
<th>( x )</th>
<th>( \frac{\sin x}{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.999 983 333</td>
<td>−0.01</td>
<td>0.999 983 333</td>
</tr>
<tr>
<td>0.001</td>
<td>0.999 999 833</td>
<td>−0.001</td>
<td>0.999 999 833</td>
</tr>
<tr>
<td>0.0001</td>
<td>0.999 999 998</td>
<td>−0.0001</td>
<td>0.999 999 998</td>
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<tr>
<td>↓</td>
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<td>↓</td>
<td>↓</td>
</tr>
<tr>
<td>0⁺</td>
<td>1</td>
<td>0⁻</td>
<td>1</td>
</tr>
</tbody>
</table>

So,

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1
\]
Graphically:

\[ f(x) = \frac{\sin x}{x} \]

\( \lim_{x \to 0} \frac{\sin x}{x} = 1 \)

Question: Are functions \( f(x) = \frac{\sin x}{x} \) and \( g(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \) the same?
Graphically:

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1
\]

\[
f(x) = \frac{\sin x}{x}
\]

**Question:** Are functions \( f(x) = \frac{\sin x}{x} \) and \( g(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \) the same?

**Answer:** No. \( f(x) \) is not defined at \( x = 0 \) but \( g(x) \) is well-defined at \( x = 0 \).
Example: Sketch the graph of \( f(x) = \begin{cases} 
 x^2 & \text{if } x > 0 \\
 -2 & \text{if } x = 0 \\
 \sqrt{1-x} & \text{if } x < 0 
\end{cases} \) and then find the following limits:

a. \( \lim_{x \to 0^-} f(x) \)  

b. \( \lim_{x \to 0^+} f(x) \)  

c. \( \lim_{x \to 0} f(x) \)
**Example:** Sketch the graph of 

\[ f(x) = \begin{cases} 
  x^2 & \text{if } x > 0 \\
  -2 & \text{if } x = 0 \\
  \sqrt{1 - x} & \text{if } x < 0 
\end{cases} \]

and then find the following limits:

\[ a. \lim_{x \to 0^-} f(x) \quad b. \lim_{x \to 0^+} f(x) \quad c. \lim_{x \to 0} f(x) \]

**Answer:**

\[ a. \lim_{x \to 0^-} f(x) = 1 \]

\[ b. \lim_{x \to 0^+} f(x) = 0 \]

\[ c. \lim_{x \to 0} f(x) \text{ DNE} \]
Example: Sketch the graph of $f(x) = \frac{|2 - x|}{x^2 - 4}$ and find the following limits:

a. $\lim_{x \to 2^-} f(x)$  
b. $\lim_{x \to 2^+} f(x)$  
c. $\lim_{x \to 2} f(x)$
Example: Sketch the graph of \( f(x) = \frac{|2-x|}{x^2 - 4} \) and find the following limits:

\[
\begin{align*}
\text{a. } \lim_{x \to 2^-} f(x) & \quad \text{b. } \lim_{x \to 2^+} f(x) \\
\text{c. } \lim_{x \to 2} f(x)
\end{align*}
\]

Answer:

\[
\frac{|2-x|}{x^2 - 4} = \begin{cases} \\
\frac{2-x}{(x-2)(x+2)} = \frac{-1}{x+2} & \text{if } x < 2 \\
\frac{-2-x}{(x-2)(x+2)} = \frac{1}{x+2} & \text{if } x > 2
\end{cases}
\]

The graph of \( f(x) = \frac{|2-x|}{x^2 - 4} \) is given below:
\[ \lim_{x \to 2^-} \frac{|2 - x|}{x^2 - 4} = -\frac{1}{4} \]
\[ \lim_{x \to 2^+} \frac{|2 - x|}{x^2 - 4} = \frac{1}{4} \]

Since \[ \lim_{x \to 2^-} \frac{|2 - x|}{x^2 - 4} \neq \lim_{x \to 2^+} \frac{|2 - x|}{x^2 - 4}, \]

\[ \lim_{x \to 2} \frac{|2 - x|}{x^2 - 4} \text{ DNE} \]