1.3 *Computation of Limits*

1. **Limits of Some Special Functions**:

i. For any **polynomial** $p(x)$ and any real number a,

$$\lim_{x\to a} p(x) = p(a).$$

ii. **Power of functions**. Let $\lim_{x\to a} f(x) = L$. Then for n a positive integer.

$$\lim_{x\to a} [f(x)]^n = L^n.$$

iii. **Root of functions**. Let $\lim_{x\to a} f(x) = L$. Then

$$\lim_{x\to a} \sqrt[n]{f(x)} = \begin{cases} \sqrt[n]{L} & \text{if } n \text{ is odd} \\ \sqrt[n]{L} & \text{if } n \text{ is even and } L \geq 0 \end{cases}.$$

iv. **Sine and Cosine functions**.

$$\begin{cases} \lim_{x\to a} \sin x = \sin a \\ \lim_{x\to a} \cos x = \cos a \end{cases}$$

v. **Exponential functions**. $\begin{cases} \lim_{x\to a} e^x = e^a \\ \lim_{x\to a} b^x = b^a \text{ for } b > 0 \text{ and } b \neq 1 \end{cases}$

vi. **Logarithmic functions**. For $a > 0$, $\begin{cases} \lim_{x\to a} \ln x = \ln a \\ \lim_{x\to a} \log_b x = \log_b a \end{cases}$

Example: *Find the following limits.*

a. $\lim\limits_{x\to 2} \sqrt{x^2 - 2x + \pi}$ b. $\lim\limits_{x\to \pi} \cos^3(x)$ c. $\lim\limits_{x\to e^2} \ln x$

a. $\lim_{x\to 2} (x^2 - 2x + \pi) = 2^2 - 2(2) + \pi = \pi > 0$

$$\lim_{x\to 2} \sqrt{x^2 - 2x + \pi} = \lim_{x\to 2} \sqrt{x^2 - 2x + \pi} = \sqrt{\pi}$$

b. $\lim_{x\to\pi} \cos(x) = -1, \ \lim_{x\to\pi} \cos^3(x) = (-1)^3 = -1.$

c. $\lim_{x\to e^2} \ln x = \ln e^2 = 2\ln e = 2.$

2. **Rules of Limits**:

Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Let c be a constant. Then

i. $\lim_{x\to a} \; c\,f(x) = c \; \lim_{x\to a} f(x)$

ii. $\lim_{x\to a}[\, f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

iii. $\lim_{x\to a}[\, f(x)g(x)] = [\lim_{x\to a} f(x)\,]\,[\,\lim_{x\to a} g(x)]$

iv. If $\lim_{x\to a} g(x) \neq 0,$ then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$.

Example: *Find the following limits algebraically.*

a. $\lim\limits_{x\to 0} \; (x^2 - 1)\, e^{-2x+1}$ b. $\lim\limits_{x\to 0} \dfrac{\sin(x+1)}{\ln(x^2+2)}$ c. $\lim\limits_{x\to 2} \; \dfrac{x^2 - 4}{x^2 + 1}$

a. $\lim_{x\to 0} \; (0^2 - 1)e^{-2(0)+1} = -e$

b. $\lim_{x\to 0} \frac{\sin(x+1)}{\ln(x^2+2)} = \frac{\sin(1)}{\ln(2)} = 1.213986017$

c. $\lim_{x\to 2} \; \frac{x^2-4}{x^2+1} = \frac{0}{5} = 0$

Question: When $\lim_{x\to a} g(x) = 0$, how can we determine if $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ exists and how can we find $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ if it exists?

Answer:

Case 1 $\lim_{x\to a} f(x) \neq 0$ $\left(\dfrac{f(x)}{g(x)} \Rightarrow \dfrac{\text{constant}}{0} \text{ as } x \to a\right)$

Then $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ DNE (Example: $\lim_{x\to 2} \dfrac{x^2+1}{x^2-4}$ DNE)

Case 2 $\lim_{x\to a} f(x) = 0$ $\left(\dfrac{f(x)}{g(x)} \Rightarrow \dfrac{0}{0} \text{ as } x \to a\right)$

Factorize both $f(x)$ and $g(x)$ by $x - a$ and simplify:

$$\frac{f(x)}{g(x)} = \frac{(x-a)\, f_1(x)}{(x-a)\, g_1(x)} = \frac{f_1(x)}{g_1(x)}$$

and $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \lim_{x\to a} \dfrac{f_1(x)}{g_1(x)}$

Example: a. $\lim_{x\to 2} \dfrac{x^2-4}{x^2-x-2}$ b. $\lim_{x\to 9} \dfrac{x-9}{3-\sqrt{x}}$ c. $\lim_{x\to 0} \dfrac{5x}{2-\sqrt{x+4}}$

a. $\lim_{x\to 2} \dfrac{x^2-4}{x^2-x-2} \overset{\frac{0}{0}}{=} \lim_{x\to 2} \dfrac{(x-2)(x+2)}{(x-2)(x+1)} = \lim_{x\to 2} \dfrac{x+2}{x+1} = \dfrac{4}{3}$

b. $\lim_{x\to 9} \dfrac{x-9}{3-\sqrt{x}} \overset{\frac{0}{0}}{=} \lim_{x\to 9} \dfrac{(x-9)(3+\sqrt{x})}{(3-\sqrt{x})(3+\sqrt{x})} = \lim_{x\to 9} \dfrac{(x-9)(3+\sqrt{x})}{9-x}$

$= \lim_{x\to 9} -(3+\sqrt{x}) = -6$

c. $\lim_{x\to 0} \dfrac{5x}{2-\sqrt{x+4}} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \dfrac{5x\left(2+\sqrt{x+4}\right)}{\left(2-\sqrt{x+4}\right)\left(2+\sqrt{x+4}\right)}$

$= \lim_{x\to 0} \dfrac{5x\left(2+\sqrt{x+4}\right)}{(4-(x+4))} = \lim_{x\to 0} \dfrac{5x\left(2+\sqrt{x+4}\right)}{-x}$

$= \lim_{x\to 0} -5\left(2+\sqrt{x+4}\right) = -20$

Example: a. $\lim_{h\to 0} \dfrac{(2+h)^3-8}{h}$ b. $\lim_{x\to 0} \dfrac{\sqrt{x^2-x+9}-3}{x^2-x}$

a.

$$\lim_{h\to 0} \frac{(2+h)^3-8}{h} = \lim_{h\to 0} \frac{8+12h+6h^2+h^3-8}{h} = \lim_{h\to 0} \frac{12h+6h^2+h^3}{h}$$

$$= \lim_{h\to 0} \frac{h(12+6h+h^2)}{h} = \lim_{h\to 0}(12+6h+h^2) = 12$$

b. $$\lim_{x\to 0} \frac{\sqrt{x^2-x+9}-3}{x^2-x} = \lim_{x\to 0} \frac{\left(\sqrt{x^2-x+9}-3\right)\left(\sqrt{x^2-x+9}+3\right)}{(x^2-x)\left(\sqrt{x^2-x+9}+3\right)}$$

$$= \lim_{x\to 0} \frac{x^2-x+9-9}{(x^2-x)\left(\sqrt{x^2-x+9}+3\right)} = \lim_{x\to 0} \frac{x^2-x}{(x^2-x)\left(\sqrt{x^2-x+9}+3\right)}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{x^2-x+9}+3} = \frac{1}{6}$$

Example: *Assume that* $\lim_{x\to 0} \dfrac{\sin(x)}{x} = 1$. *Find the following limits algebraically.*

a. $\lim\limits_{x\to 0} x^2 \csc^2 x$ b. $\lim\limits_{x\to 0} \dfrac{\sin(3x)}{2x}$ c. $\lim\limits_{x\to 0} \dfrac{\tan(3x)}{2x}$

a. $\lim_{x\to 0} x^2 \csc^2 x = \lim_{x\to 0} \dfrac{x^2}{\sin^2 x} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\left(\dfrac{x}{\sin x}\right)^2 = \left[\lim_{x\to 0} \dfrac{x}{\sin x}\right]^2$

Because

$$\lim_{x\to 0} \frac{x}{\sin x} = \lim_{x\to 0} \frac{1}{\frac{\sin x}{x}} = \frac{1}{1} = 1, \quad \left[\lim_{x\to 0} \frac{x}{\sin x}\right]^2 = 1^2 = 1$$

b. $\lim_{x\to 0} \dfrac{\sin(3x)}{2x} = \lim_{x\to 0}\left(\frac{1}{2}\right)(3)\dfrac{\sin(3x)}{3x} = \frac{3}{2}\lim_{x\to 0} \dfrac{\sin(3x)}{3x} = \frac{3}{2}$

c. $\lim_{x\to 0} \dfrac{\tan(3x)}{2x} = \lim_{x\to 0} \dfrac{\sin(3x)}{2x\cos(3x)} = \lim_{x\to 0} \dfrac{\sin(3x)}{2x}\left(\dfrac{1}{\cos(3x)}\right)$

$$= \left(\frac{3}{2}\right)\frac{1}{1} = \frac{3}{2}$$

3. **Squeeze Theorem (Sandwich Theorem)**:

Theorem: Suppose that $f(x) \le g(x) \le h(x)$ for all x in (a, b), except possible at the point c containing in (a, b), and

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L.$$

Then

$$\lim_{x \to c} g(x) = L.$$

Example: Use **the Squeeze Theorem** to find $\lim_{x \to 0}\left(x^2 \cos\left(\frac{1}{x}\right)\right)$.

Observation: as $x \to 0$, $\frac{1}{x} \to \pm\infty$ and values of $\cos\left(\frac{1}{x}\right)$ are oscillated but are between -1 and 1. Since, $x^2 \to 0$, we **know** the limit is 0. Now we show this **algebraically**.

It is known that $-1 \le \cos\left(\frac{1}{x}\right) \le 1 \quad \overset{\text{multiple each side by } x^2}{\Longleftrightarrow} \quad -x^2 \le x^2 \cos\left(\frac{1}{x}\right)$

Since ($f(x) = -x^2$, $g(x) = x^2 \cos\left(\frac{1}{x}\right)$, $h(x) = x^2$)

$$\lim_{x \to 0}(-x^2) = 0 \text{ and } \lim_{x \to 0}(x^2) = 0 \Rightarrow \lim_{x \to 0}\left(x^2 \cos\left(\frac{1}{x}\right)\right) = 0.$$