1.4 Continuity and Its Consequences

Graphically, a function $f$ is **continuous** at $x = a$ if the graph of $f$ has no break up (i.e., hole, vertical asymptote, jump) at $x = a$.

**Examples:** Functions that are not continuous at $x = 0$, or $x = 2$.

\[ f(x) = \frac{\sin x}{x} \quad \text{a hole at } x = 0 \]

\[ f(x) = \frac{x^2 - 5}{x - 2}, \quad \text{vertical asymptote at } x = 2 \]

\[ f(x) = \frac{|2 - x|}{x^2 - 4} \quad \text{a jump at } x = 2 \]
**Definition:** A function $f$ is **continuous** at $x = a$ if

i. $f(a)$ is defined;

ii. $\lim_{x \to a} f(x)$ exists; and

iii. $\lim_{x \to a} f(x) = f(a)$.

Otherwise, $f$ is said to be **discontinuous** at $x = a$.

**Note that**

- $f$ is **continuous** at $x = a$ if and only if **all three** (i., ii., iii.) conditions are satisfied.
- $f$ is **not** continuous at $x = a$ if **one** of these three conditions does **not** hold.
Use the **definition** of continuity to explain **the break up** of the graph of $f$ at $x = a$:

- **hole** -
  - $f(a)$ is not defined but $\lim_{x\to a} f(x) = L$
    - the condition I. does not hold
  - $f(a)$ is defined and $\lim_{x\to a} f(x) = L$ but $L \neq f(a)$
    - the condition III. does not hold

- **vertical asymptote** -
  - $f(a)$ is not defined and $\lim_{x\to a^-} f(x) = \infty$ or $-\infty$ or $\lim_{x\to a^+} f(x) = \infty$ or $-\infty$
    - the condition I. does not hold

- **jump** -
  - $\lim_{x\to a^-} f(x) = L_1$, $\lim_{x\to a^+} f(x) = L_2$ but $L_1 \neq L_2$
    - the condition II. does not hold
Example: Page 106: 1-3:

1. $f$ is not continuous at $x = -2$ because

$$\lim_{x \to -2} f(x) \text{ DNE } (\lim_{x \to -2} f(x) = 2 \text{ and } \lim_{x \to -2^+} f(x) = -1).$$

2. $f$ is not continuous at $x = -2$ because

$$\lim_{x \to -2} f(x) \neq f(-2) \text{ (} \lim_{x \to -2} f(x) = 3 \text{ and } f(-2) = -2).$$

3. $f$ is not continuous at $x = -2$ because $f(-2)$ is not defined (or $\lim_{x \to -2} f(x) \text{ DNE}$).
Example: Determine by the definition of continuity if \( f \) is continuous at \( x = 2 \). Provide a reason for each answer.

a. \( f(x) = \frac{x^2 - 4}{x - 2} \)  

b. \( f(x) = \begin{cases} 
\frac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\
4 & \text{if } x = 2 
\end{cases} \)

c. \( f(x) = \begin{cases} 
\frac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\
3 & \text{if } x = 2 
\end{cases} \)

a. \( f \) is not continuous at \( x = 2 \) because \( f(x) \) is not defined at \( x = 2 \).
b. \( f(2) = 4 \). Now check to see if \( \lim_{x \to 2} f(x) = 4 \). Check:

\[
\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4
\]

Hence, \( f \) is continuous at \( x = 2 \) by definition.
c. \( f(2) = 3 \). \( f \) is not continuous at \( x = 2 \) because from b. \( \lim_{x \to 2} f(x) = 4 \neq 3 \).
Discontinuous points and removable discontinuous points:

Suppose that \( \lim_{x \to a} f(x) = L \) and \( f(a) \) is either not defined or \( f(a) \neq L \). Then \( x = a \) is called a **removable** discontinuous point of \( f(x) \) in the sense that the new function \( g(x) = \begin{cases} f(x) & \text{if } x \neq a \\ L & \text{if } x = a \end{cases} \) is continuous at \( x = a \).

Note: If \( f(x) \) is not defined at \( x = a \) but \( \lim_{x \to a} f(x) = L \), then \( x = a \) is a removable discontinuous point of \( f(x) \).

**Example:** The graph of \( f(x) \) is given below. From the graph, we know \( f(x) \) is not continuous at (i) \( x = -1 \), (ii) \( x = 0 \), (iii) \( x = -1 \) and (iv) \( x = 2.5 \). For each point, specify a **continuity condition** (given in the definition of continuity) that fails to satisfy and determine if it is a removable discontinuous point.
Answer:

(i) \(f(x)\) is not continuous at \(x = -1\) because \(f(-1)\) is not defined. Since \(\lim_{x \to -1} f(x) = 1.5\), \(x = -1\) is a removable continuous point.

(ii) \(f(x)\) is not continuous at \(x = 0\) because \(\lim_{x \to 0} f(x)\) DNE and therefore \(x = 0\) is not a removable discontinuous point.

(iii) \(f(x)\) is not continuous at \(x = 1\) because \(\lim_{x \to 1} f(x) \neq f(1)\). Since \(\lim_{x \to 1} f(x) = 1\), \(x = 1\) is a removable continuous point.

(iv) \(f(x)\) is not continuous at \(x = 2.5\) because \(\lim_{x \to 0} f(x)\) DNE (\(f(2.5)\) is not defined) and therefore \(x = 2.5\) is not a removable discontinuous point.
Continuity of Special Functions:

These functions are **continuous** everywhere in their domains.

1. **All polynomials** are continuous everywhere
   (that is, they are continuous for $x$ in $(-\infty, \infty)$)

2. Functions $e^x$, $\sin x$, $\cos x$ are continuous everywhere.

3. Function $\sqrt[n]{f(x)}$ where $n$ is an **even integer** is continuous wherever $f(x) \geq 0$.

4. Function $\ln[f(x)]$ is **continuous** wherever $f(x) > 0$.

**Rules:** Suppose that $f$ and $g$ are **continuous** at $x = a$. Then

1. $cf$ is continuous at $x = a$
2. $f \pm g$ is continuous at $x = a$
3. $fg$ is continuous at $x = a$
4. $\frac{f}{g}$ is continuous at $x = a$ if $g(a) \neq 0$ and is discontinuous at $x = a$ if $g(a) = 0$
A function $f$ is said to be continuous on an open interval $(a, b)$ if $f$ is continuous at every point in $(a, b)$.

A function $f$ is said to be continuous on a closed interval $[a, b]$ if $f$ is continuous on $(a, b)$ and

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b).$$

A function $f$ is said to be continuous everywhere if $f$ is continuous at every point in $(-\infty, \infty)$. 

Continuity on a Closed Interval:

- A function $f$ is said to be continuous on an open interval $(a, b)$ if $f$ is continuous at every point in $(a, b)$.

- A function $f$ is said to be continuous on a closed interval $[a, b]$ if $f$ is continuous on $(a, b)$ and

  $$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b).$$

- A function $f$ is said to be continuous everywhere if $f$ is continuous at every point in $(-\infty, \infty)$. 


**Example:** Determine the intervals on which $f(x)$ is continuous.

a. $f(x) = 2x^2 - x + 2$  
   b. $f(x) = 2\sin(3x) - 5\cos(3x)$  
   c. $f(x) = 3e^{-x/2}$  
   d. $f(x) = \sqrt{2 + 3x}$  
   e. $f(x) = \sqrt{x^2 - 2}$  
   f. $f(x) = (2x + 1)^{5/2}$  
   g. $f(x) = \ln(1 - 2x)$  
   h. $f(x) = \ln(x^2 + x - 2)$  
   i. $f(x) = \frac{x}{x^2 - 2}$  
   j. $f(x) = \frac{x - 1}{x^2 - x - 2}$

**Answers:** These functions are continuous on their **domains**. So, we will first find their domains.

a. $f(x) = 2x^2 - x + 2$ is a polynomial and its domain is $(-\infty, \infty)$.  
   $f(x)$ is continuous on $(-\infty, \infty)$.

b. $f(x) = 2\sin(3x) - 5\cos(3x)$. Domains of $\sin(3x)$ and $\cos(3x)$ are $(-\infty, \infty)$. So, $f(x)$ is continuous on $(-\infty, \infty)$. 

c. \( f(x) = 3e^{-x^2} \). Domain of \( 3e^{-x^2} \) is \((-\infty, \infty)\). So, \( f(x) \) is continuous on \((-\infty, \infty)\).

d. \( f(x) = \sqrt{2 + 3x} \). The domain of \( f(x) \):

\[
2 + 3x \geq 0 \iff 3x \geq -2 \iff x \geq -\frac{2}{3}
\]

\[
D_f = \{ x : 2 + 3x \geq 0 \} = \{ x : x \geq -\frac{2}{3} \} = \left[ -\frac{2}{3}, \infty \right)
\]

e. \( f(x) = \sqrt{x^2 - 2} \). The domain of \( f(x) \):

\[
x^2 - 2 \geq 0 \iff x^2 \geq 2 \iff |x| \geq \sqrt{2} \iff \begin{cases} x \geq \sqrt{2} \\
 \text{or} \\
 x \leq -\sqrt{2}
\end{cases}
\]

\[
D_f = \{ x : x^2 - 2 \geq 0 \} = \{ x : x \geq \sqrt{2} \text{ or } x \leq -\sqrt{2} \}
\]

Hence, \( f \) is continuous on \((-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)\).
\( f(x) = (2x + 1)^{5/2} \). Note that \( f(x) = (2x + 1)^{5/2} = \sqrt{(2x + 1)^5} \).

The domain of \( f(x) \):

\[
D_f = \{ x : 2x + 1 \geq 0 \} = \{ x : x \geq -\frac{1}{2} \}.
\]

Hence, \( f \) is continuous on \([-\frac{1}{2}, \infty)\).

\( g. f(x) = \ln(1 - 2x). \) The domain of \( f(x) \):

\[
1 - 2x > 0 \quad \iff \quad 2x < 1 \quad \iff \quad x < \frac{1}{2}
\]

\[
D_f = \{ x : 1 - 2x > 0 \} = \{ x : x < \frac{1}{2} \} = \left( -\infty, \frac{1}{2} \right)
\]

Hence, \( f \) is continuous on \( \left( -\infty, \frac{1}{2} \right) \).
h. \( f(x) = \ln(x^2 + x - 2) \). The domain of 
\[ f(x) : D_f = \{ x : x^2 + x - 2 > 0 \} \]

\[ x^2 + x - 2 = (x + 2)(x - 1) > 0 \]

Ways to solve the inequalities: \((x + 2)(x - 1) > 0\)

**Graphically,** the graph of \((x + 2)(x - 1)\)

(i) is shown right.

\((x + 2)(x - 1) > 0 \iff x < -2, \text{ or } x > 1.\)

(ii) **Algebraically,** let \( g(x) = (x + 2)(x - 1) \). Check

\[ g(-3) = (-1)(-4) = 4 > 0, \quad g(0) = (2)(-1) = -2, \quad g(2) = (3)(1) = 3 > 0 \]

So, \((x + 2)(x - 1) > 0 \iff x < -2, \text{ or } x > 1. \quad \text{Hence,} \]

\[ D_f = (-\infty, -2) \cup (1, \infty) \text{ and } f \text{ is continuous on } (-\infty, -2) \cup (1, \infty). \]
i. \( f(x) = \frac{x}{x^2 - 2} \). The domain of \( f(x) \) :
\[
x^2 - 2 \neq 0 \iff x^2 \neq 2 \iff x \neq \pm \sqrt{2}
\]
\[
D_f = \{ x : x \neq \pm \sqrt{2} \} = (-\infty, -\sqrt{2}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{2}, \infty).
\]
Hence, \( f \) is continuous on \( (-\infty, -\sqrt{2}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{2}, \infty) \).

j. \( f(x) = \frac{x - 1}{x^2 - x - 2} \). The domain of \( f(x) \) :
\[
x^2 - x - 2 \neq 0 \iff (x - 2)(x + 1) \neq 0 \iff x \neq 2, -1
\]
\[
D_f = \{ x : x \neq 2, -1 \} = (-\infty, -1) \cup (-1, 2) \cup (2, \infty).
\]
Hence, \( f \) is continuous on \( (-\infty, -1) \cup (-1, 2) \cup (2, \infty) \).