1.5 Limits Involving Infinity

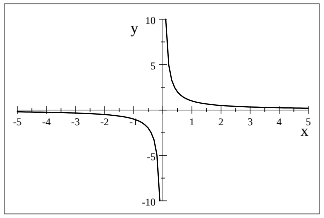
1. **Infinity as a Limit**: $\lim_{x\to a} f(x) = \infty, \lim_{x\to a} f(x) = -\infty$

Consider limits:

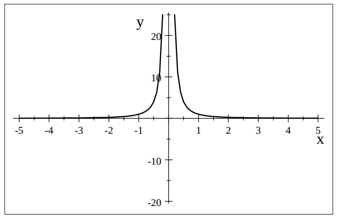
$$\lim_{x\to 0^-}\frac{1}{x}, \qquad \lim_{x\to 0^+}\frac{1}{x}, \qquad \lim_{x\to 0^-}\frac{1}{x^2}, \qquad \lim_{x\to 0^+}\frac{1}{x^2}$$

Graphs of $\frac{1}{x}$ and $\frac{1}{x^2}$:

$y = \frac{1}{x}$



$y = \frac{1}{x^2}$



Results:

$$\lim_{x\to 0^-}\frac{1}{x} \overset{\frac{1}{-0}}{=} -\infty, \quad \lim_{x\to 0^+}\frac{1}{x} \overset{\frac{1}{+0}}{=} \infty, \quad \lim_{x\to 0^-}\frac{1}{x^2} = \lim_{x\to 0^+}\frac{1}{x^2} \overset{\frac{1}{+0}}{=} \infty.$$

Results for functions $f(x) = \frac{1}{x^n}$ where n is a **positive integer**:

n is **odd**:

$$\begin{cases} \lim_{x\to 0^-}\frac{1}{x^n} = -\infty \\ \lim_{x\to 0^+}\frac{1}{x^n} = \infty \end{cases}$$

;

n is **even**:

$$\begin{cases} \lim_{x\to 0^-}\frac{1}{x^n} = \infty \\ \lim_{x\to 0^+}\frac{1}{x^n} = \infty \end{cases}$$

Results for functions $f(x) = \frac{1}{(x-a)^n}$ where n is a positive integer:

n is **odd**:

$$\begin{cases} \lim_{x\to a^-}\frac{1}{(x-a)^n} = -\infty \\ \lim_{x\to a^+}\frac{1}{(x-a)^n} = \infty \end{cases}$$

;

n is **even**:

$$\begin{cases} \lim_{x\to a^-}\frac{1}{(x-a)^n} = \infty \\ \lim_{x\to a^+}\frac{1}{(x-a)^n} = \infty \end{cases}$$

Definition of a vertical asymptote:

The vertical line $x = \alpha$ is a **vertical asymptote** of the graph of $f(x)$ if $f(x)$ is defined for $x < \alpha$ and $\lim_{x\to\alpha^-} f(x) = \infty$ or $-\infty$; or if f is defined for $x > \alpha$ and $\lim_{x\to\alpha^+} f(x) = \infty$ or $-\infty$.

Example: *Determine if each of the following limits is a constant, ∞, $-\infty$, or DNE.*

a. $\lim_{x\to 1^-}\frac{1-2x}{x^2-1}$ *b.* $\lim_{x\to -1^-}\frac{1-2x}{x^2-1}$ *c.* $\lim_{x\to -3^-}\frac{x-1}{(x+3)^2}$

d. $\lim_{x\to 2^-}\frac{x+1}{x^2-5x+6}$ *e.* $\lim_{x\to 1^-}\frac{2-x}{\sqrt{1-x^2}}$ *f.* $\lim_{x\to 1^+}\frac{2-x}{\sqrt{1-x^2}}$

g. $\lim_{x\to 0^+}\ln(5x)$ *h.* $\lim_{x\to 0^-}\ln(5x)$ *i.* $\lim_{x\to +\infty} e^{-x}\sin(\pi x)$

a. $\lim_{x\to 1^-}\frac{1-2x}{x^2-1} = \lim_{x\to 1^-}\frac{1-2x}{(x-1)(x+1)} \underset{x-1<0}{\overset{x<1}{=}} \left(\frac{-1}{(-0)(2)}\right) = \infty$

b. $\lim_{x\to -1^-}\frac{1-2x}{x^2-1} = \lim_{x\to -1^-}\frac{1-2x}{(x-1)(x+1)} \underset{x+1<0}{\overset{x<-1}{=}} \left(\frac{-1}{(-2)(-0)}\right) = -\infty$

c. $\lim_{x\to -3^-}\frac{x-1}{(x+3)^2} \underset{(x+3)^2>0}{\overset{x+3<0}{=}} \left(\frac{-4}{+0}\right) = -\infty$

d. $\lim_{x\to 2^-}\frac{x+1}{x^2-5x+6} = \lim_{x\to 2^-}\frac{x+1}{(x-2)(x-3)} \underset{x-2<0}{\overset{x<2}{=}} \left(\frac{3}{(-0)(-1)}\right) = \infty$

e. $\lim_{x\to 1^-}\frac{2-x}{\sqrt{1-x^2}} = \lim_{x\to 1^-}\frac{2-x}{\sqrt{(1-x)(1+x)}} \underset{1-x>0}{\overset{x<1}{=}} \left(\frac{1}{\sqrt{(+0)(2)}}\right) = \infty$

f. $\lim_{x\to 1^+} \frac{2-x}{\sqrt{1-x^2}} = \lim_{x\to 1^+} \frac{2-x}{\sqrt{(1-x)(1+x)}} \overset{x>1}{\underset{1-x<0}{=}} \left(\frac{1}{\sqrt{(-0)(2)}} \right)$ =DNE

g. $\lim_{x\to 0^+} \ln(5x) = \ln(+0) = -\infty$

h. $\lim_{x\to 0^-} \ln(5x) \overset{x<0}{\underset{5x<0}{=}} \ln(-0) =$ DNE

i. $\lim_{x\to +\infty} e^{-x}\sin(\pi x) = 0$ ($\lim_{x\to +\infty} e^{-x} = 0$ and $-1 \le \sin(\pi x) \le 1$)

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2. **Limits at Infinity**: $\lim_{x\to\infty} f(x)$, $\lim_{x\to -\infty} f(x)$

We study the behaviors of a function $f(x)$ as $x \to \infty$ and $x \to -\infty$.

This concept is not new to us!!

Recall: $y = L$ is said to be a **horizontal asymptote** of the graph of f if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to -\infty} f(x) = L$.

Results:

$$\lim_{x\to\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to -\infty} \frac{1}{x} = 0.$$

For any positive integer n,

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \quad \text{and} \quad \lim_{x\to -\infty} \frac{1}{x^n} = 0.$$

For any positive rational number $n = \frac{p}{q}$:

$$\lim_{x\to\infty} \frac{1}{x^{p/q}} = 0, \quad \text{and} \quad \lim_{x\to -\infty} \frac{1}{x^{p/q}} = 0 \quad (q \text{ is odd}).$$

Horizontal asymptote of a **rational function**:

Let

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

where m and n are positive integers.

Consider limits:

$$\lim_{x\to\infty} f(x) \quad \text{and} \quad \lim_{x\to -\infty} f(x).$$

Observe that as $x \to \infty$ or $x \to -\infty$,

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \overset{\text{same behavior}}{\approx} a_n x^n$$

and

$$b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0 \overset{\text{same behavior}}{\approx} b_m x^m.$$

Hence,

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m} .$$

Three cases: $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m}$

i. $m = n$, then

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_n x^n} = \lim_{x\to\pm\infty} \frac{a_n}{b_n} = \frac{a_n}{b_n}.$$

Hence, the graph of $f(x)$ has a horizontal asymptote: $y = \frac{a_n}{b_n}$.

ii. $m > n$, then

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m} = \lim_{x\to\pm\infty} \frac{a_n}{b_m}\left(\frac{1}{x^{m-n}}\right) = \frac{a_n}{b_m}(0) = 0.$$

Hence, the graph of $f(x)$ has a horizontal asymptote: $y = 0$.

iii. $m < n$, then

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m} = \lim_{x\to\pm\infty} \frac{a_n}{b_m}(x^{n-m}) = \frac{a_n}{b_m}(\pm\infty) = \infty \text{ or } -\infty$$

Hence, the graph of f **does not** have a horizontal asymptote.

In summary,

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0}{b_mx^m + b_{m-1}x^{m-1} + \cdots + b_1x + b_0}$$

$$= \lim_{x\to\pm\infty} \frac{a_nx^n}{b_mx^m} = \begin{cases} \text{i. } \dfrac{a_n}{b_n} & \text{if } m = n \\ \text{ii. } 0 & \text{if } m > n \\ \text{iii. } \infty \text{ or } -\infty & \text{if } m < n \end{cases}.$$

that is, the graph of $f(x)$ has

i. a horizontal asymptote $y = \frac{a_n}{b_n}$ if $m = n$;

ii. a horizontal asymptote $y = 0$ if $m > n$; or

iii. no horizontal asymptote if $m < n$.

Example: *Find the horizontal asymptote(s) and vertical asymptote(s) of the graph of* f *if they exist.*

$a.\ f(x) = \dfrac{2x^2 - x + 1}{1 - 3x^2}$ $\quad b.\ f(x) = \dfrac{x}{3 - x^2}$ $\quad c.\ f(x) = \dfrac{x^3 + 1}{x^2 + 2}$

a. $f(x) = \dfrac{2x^2 - x + 1}{1 - 3x^2}$, in this case, $m = n = 2,\ a_2 = 2,\ b_2 = -3,$ so

$$\lim_{x\to\pm\infty} \frac{2x^2 - x + 1}{1 - 3x^2} = \frac{2}{(-3)} = -\frac{2}{3}$$

Horizontal asymptote: $y = -\frac{2}{3}$.

Vertical asymptote: Set $1 - 3x^2 = 0$ and solve for x :

$3x^2 = 1,\quad x^2 = \frac{1}{3},\quad x = \pm\frac{1}{\sqrt{3}},$ vertical asymptotes: $x = \pm\frac{1}{\sqrt{3}}$

b. $f(x) = \dfrac{x}{3 - x^2},$ in this case $m = 2 > n = 1,$ so $\lim_{x\to\pm\infty} \dfrac{x}{3 - x^2} = 0.$

Horizontal asymptote: $y = 0$

Vertical asymptote: Set $3 - x^2 = 0.\quad x^2 = 3,\quad x = \pm\sqrt{3},$ vertical asymptotes: $x = \pm\sqrt{3}$.

c. $f(x) = \dfrac{x^3 + 1}{x^2 + 2},$ in this case $n = 3 > m = 2,$ so $\lim_{x\to\pm\infty} \dfrac{x^3 + 1}{x^2 + 2} = \pm\infty$ and the graph does not have a horizontal asymptote. Since $x^2 + 2 \neq 0,$ there is no vertical asymptote.

Example: *Find the following limits.*

$a.\ \lim_{x\to\infty} \dfrac{2 - x^2}{\sqrt{x^4 + x - 1}}$ $\quad b.\ \lim_{x\to\infty}\left(\sqrt{4x^2 - 5x + 17} - 2x\right)$

Note that $f(x)$ in these two examples are not rational functions. Though we cannot use the results given above directly, we can use the same idea to derive the limits.

a. $\lim_{x\to\infty} \dfrac{2 - x^2}{\sqrt{x^4 + x - 1}}$

This is a $\frac{-\infty}{\infty}$ type. Observe that as $x \to \infty$ or $-\infty$, $\sqrt{x^4 + x - 1}$ is dominated by $\sqrt{x^4} = x^2$ and $2 - x^2$ is dominated by $-x^2$. So,

$$\lim_{x\to\infty} \frac{2 - x^2}{\sqrt{x^4 + x - 1}} = \lim_{x\to\infty} \frac{-x^2}{x^2} = \frac{-1}{1} = -1.$$

b. $\lim_{x\to\infty}\left(\sqrt{4x^2-5x+17}-2x\right)$

It is a $\infty - \infty$ type. First we need to rewrite the function. Recall the identity:

$$a^2 - b^2 = (a-b)(a+b).$$

The factor $(a+b)$ is called the factor conjugate to $(a-b)$ ($(a-b)$ is also called the factor conjugate to $(a+b)$). Factors $(a+b)$ and $(a-b)$ are called a conjugate pair. Let us consider

$$a = \sqrt{4x^2-5x+17} \quad \text{and} \quad b = 2x.$$

Then $\sqrt{4x^2-5x+17}-2x$ is of the form $a-b$ and the factor conjugate to it is

$$a+b = \sqrt{4x^2-5x+17}+2x.$$

$$\lim_{x\to\infty}\left(\sqrt{4x^2-5x+17}-2x\right)$$

$$= \lim_{x\to\infty}\frac{\left(\sqrt{4x^2-5x+17}-2x\right)\left(\sqrt{4x^2-5x+17}+2x\right)}{\sqrt{4x^2-5x+17}+2x}$$

$$= \lim_{x\to\infty}\frac{\left(\sqrt{4x^2-5x+17}\right)^2-(2x)^2}{\sqrt{4x^2-5x+17}+2x} = \lim_{x\to\infty}\frac{4x^2-5x+17-(2x)^2}{\sqrt{4x^2-5x+17}+2x}$$

$$= \lim_{x\to\infty}\frac{-5x+17}{\sqrt{4x^2-5x+17}+2x} = \lim_{x\to\infty}\frac{-5x}{\sqrt{4x^2}+2x} = \lim_{x\to\infty}\frac{-5x}{2|x|+2x}$$

$$= \lim_{x\to\infty}\frac{-5x}{2x+2x} = \lim_{x\to\infty}\frac{-5x}{4x} = \frac{-5}{4}$$

Example: *Suppose that the length of a small animal* t *days after its birth is given by*

$$L(t) = \frac{100}{2+3(0.4)^t} \text{ mm.}$$

What is the length of the animal at birth? What is the eventual length of the animal (i.e., the length as $t \to \infty$).

The length of the animal at birth is

$$L(0) = \frac{100}{2+3(0.4)^0} = \frac{100}{2+3(1)} = 20 \text{ mm.}$$

The eventual length of the animal is

$\lim_{t\to\infty} L(t) = \lim_{t\to\infty}\frac{100}{2+3(0.4)^t}$. Observe that $\lim_{t\to\infty}(0.4)^t = 0$. So

$$\lim_{t\to\infty} L(t) = \lim_{t\to\infty}\frac{100}{2+3(0.4)^t} = \frac{100}{2+3(0)} = 50 \text{ mm.}$$

3. Slant Asymptotes (Oblique Asymptotes):

As we derived previously, in the case where $m < n$, the graph of f does not have a horizontal asymptote. Observe that in this case by Long Division $f(x)$ can be expressed as of the form:

$$f(x) = p(x) + \frac{R(x)}{Q(x)}$$

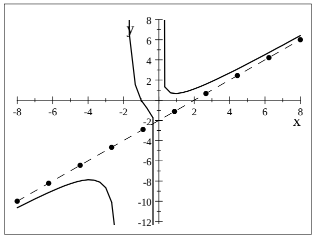
where $p(x)$ is a polynomial of degree $n-m$, and $R(x)$ and $Q(x)$ are also polynomials and the degree of $R(x)$ is less than the degree of $Q(x)$. As $x \to \pm\infty$, $\frac{R(x)}{Q(x)} \to 0$ so the behavior of $f(x)$ is very much like the one of $p(x)$. When $p(x)$ is **a linear function** ($n = m+1$), $y = p(x)$ is called a **slant asymptote** of the graph of f.

Example: *Find a slant asymptote of* $f(x) = \frac{x^3+1}{x^2+2x}$.

Since $m = 2$ and $n = 3$, the graph of f has a slant asymptote. Apply Long Division:

$$\begin{array}{r l}
 & x-2 \\
x^2+2x \,| & \overline{x^3+0x^2+0x+1} \\
 & \underline{x^3+2x^2} \\
 & \quad -2x^2\ +0x\ +1 \\
 & \quad \underline{-2x^2-4x} \\
 & \qquad\qquad 4x+1
\end{array}$$

Graphically,



- $y = x + \dfrac{-2x+1}{x^2+2x}$,

$f(x) = x-2+\dfrac{4x+1}{x^2+2x}$, the slant asymptote is $y = x-2$