2.1 Tangent Lines and Velocity

1. **Slope of a Line**:

Let L be a line passing through points (x_1, y_1) and (x_2, y_2). Then

the **slope** of L	$m = \dfrac{y_2 - y_1}{x_2 - x_1}$
the **equation** of L	$y - y_1 = m(x - x_1)$

If these two points are on a curve $y = f(x)$, then this line is also called a **secant line** to the curve.

2. **Difference Quotient and Secant Lines**:

Consider a curve $y = f(x)$. Let $h > 0$. The **slope** m_{sec} **of the secant line** through the points $(a,\ f(a))$ and $(a+h,\ f(a+h))$ is

$$m = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

m is also called a **difference quotient**.

3. **Slope of a Tangent Line**:

The **tangent line** to the curve $y = f(x)$ at $x = a$ is the line that **touches** the curve at only **one point** $(a,\ f(a))$ when x is near a. The **slope** $m_{\tan}$ **of the tangent line** to the curve $y = f(x)$ at $x = a$ is defined as

$$m_{\tan} = \lim_{h \to 0} m_{\sec} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists. The **equation of the tangent line** at the point $(a,\ f(a))$ is

$$y - f(a) = m_{\tan}\ (x - a) \qquad \text{or} \qquad y = m_{\tan}\ (x - a) + f(a)$$

Steps for computing $m_{\tan}$:

(**i**) Compute $f(a)$.

(**ii**) Compute and simplify the difference quotient: $\dfrac{f(a+h) - f(a)}{h}$.

iii. Compute $m_{\tan} = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ if the limit exists.

Example: *Find the equation of the tangent line to the curve* $y = 2x^3 - x$ *at* $x = -1$.

(i) $f(-1) = 2(-1)^3 - (-1) = -1$

(ii) Compute and simplify the difference quotient:

$$\frac{f(-1+h) - f(-1)}{h} = \frac{2(-1+h)^3 - (-1+h) - (-1)}{h}$$

$$= \frac{2(h^3 - 3h^2 + 3h - 1) - h + 1 + 1}{h}$$

$$= \frac{2h^3 - 6h^2 + 5h}{h} = 2h^2 - 6h + 5$$

(iii) Compute $m_{\tan}$

$$m_{\tan} = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \to 0}(2h^2 - 6h + 5) = 5$$

(iv) The equation of the tangent line:

$$y - (-1) = 5(x - (-1)) \Rightarrow y = 5x + 5 - 1 \Rightarrow y = 5x + 4$$

Example: *Find the equation of the tangent line to the curve* $y = \frac{3x}{x+1}$ *at* $x = 1$.

(i) $f(1) = \frac{3(1)}{(1)+1} = \frac{3}{2}$.

(ii) Compute and simplify the difference quotient:

$$\frac{f(1+h)-f(1)}{h} = \frac{\frac{3(1+h)}{(1+h)+1} - \frac{3}{2}}{h} = \frac{\frac{3(1+h)}{2+h} - \frac{3}{2}}{h}$$

$$= \frac{\frac{6+6h-6-3h}{2(2+h)}}{h} = \frac{\frac{3h}{2(2+h)}}{h} = \frac{3}{2(2+h)}$$

(iii) Compute $m_{\tan} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{3}{2(2+h)} = \frac{3}{4}$

(iv) The equation of the tangent line:

$$y - \frac{3}{2} = \frac{3}{4}(x-1) \Rightarrow y = \frac{3}{4}x - \frac{3}{4} + \frac{3}{2} = \frac{3}{4}x + \frac{3}{4}$$

Example: *Find the equation of the tangent line to the curve* $y = \sqrt{2x-3}$ *at* $x = 2$.

(i) $f(2) = \sqrt{2(2)-3} = \sqrt{1} = 1.$

(ii) Compute and simplify the difference quotient:

$$\frac{f(2+h)-f(2)}{h} = \frac{\sqrt{2(2+h)-3}-1}{h} = \frac{\sqrt{2h+1}-1}{h}$$

$$= \frac{\left(\sqrt{2h+1}-1\right)\left(\sqrt{2h+1}+1\right)}{h\left(\sqrt{2h+1}+1\right)} = \frac{2h+1-1}{h\left(\sqrt{2h+1}+1\right)}$$

$$= \frac{2h}{h\left(\sqrt{2h+1}+1\right)} = \frac{2}{\sqrt{2h+1}+1}$$

(iii) Compute $m_{\text{tan}} = \lim_{h\to 0} \dfrac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \dfrac{2}{\sqrt{2h+1}+1} = 1$

(iv) The equation of the tangent line:

$$y - 1 = (1)(x-2) \Rightarrow y = x - 2 + 1 = x - 1$$

4. **<u>Velocity and Instantaneous Rate of Change</u>**: If $f(t)$ represents the **<u>position of an object</u>** at time t, then

- $\frac{f(a+h)-f(a)}{h}$ = **average velocity** of $f(t)$ in $[a,\ a+h]$
- $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ = **velocity of** $f(t)$ at $t = a$

  instantaneous velocity = **velocity**

Example: *Let* $f(t) = 4t^2 - 3t + 1$ *be a distance function in feet of a moving object at time* t *in seconds. Find the velocity of the object when* $t = 1$ *second.*

(i) $f(1) = 4 - 3 + 1 = 2$

(ii) Compute and simplify the difference quotient:

$$\frac{f(1+h) - f(1)}{h} = \frac{4(1+h)^2 - 3(1+h) + 1 - 2}{h}$$

$$= \frac{4(1 + 2h + h^2) - 3 - 3h - 1}{h} = \frac{4 + 8h + 4h^2 - 3h - 4}{h}$$

$$= \frac{5h + 4h^2}{h} = \frac{h(5 + 4h)}{h} = 5 + 4h$$

(iii) Compute $v(1)$

$$v(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0}(5 + 4h) = 5 \text{ ft/sec}$$

Example: *Suppose that the population of a city is estimated to be* $f(t) = \sqrt{100+8t}$ *million people* t *years from now. Find the rate of change of the population 2 years from now.*

(i) $f(2) = \sqrt{100+16} = \sqrt{116}$

(ii) Compute and simplify the difference quotient:

$$\frac{f(2+h)-f(2)}{h} = \frac{\sqrt{100+8(2+h)} - \sqrt{116}}{h} = \frac{\sqrt{116+8h} - \sqrt{116}}{h}$$

$$= \frac{\left(\sqrt{116+8h} - \sqrt{116}\right)\left(\sqrt{116+8h} + \sqrt{116}\right)}{h\left(\sqrt{116+8h} + \sqrt{116}\right)}$$

$$= \frac{116+8h-116}{h\left(\sqrt{116+8h} + \sqrt{116}\right)} = \frac{8}{\sqrt{116+8h} + \sqrt{116}}$$

(iii) the instantaneous rate of change:

$$\lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{8}{\sqrt{116+8h} + \sqrt{116}} = \frac{8}{2\sqrt{116}} = 0.3714 \text{ million}$$

That means that the population of this city grows 0.3714 million people per year two years