2.2 Derivatives

1. Definition of Derivative at a Point:
The derivative of the function \( f(x) \) at \( x = a \) is defined as
\[
f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]
provided the limit exists.

If the limit exists, we say that \( f \) is differentiable at \( x = a \), otherwise, we say that \( f \) is not differentiable at \( x = a \). An alternative form of \( f'(a) \):
\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]
provided the limit exists.

Remarks:
\[
f'(a) = \begin{cases} 
\text{slope of tangent to } y = f(x) \text{ at } (a, f(a)) \\
\text{rate of change of } f(x) \text{ at } x = a \\
\text{velocity of the object at } x = a \text{ when } f \text{ is a position function}
\end{cases}
\]

Example: Find the derivative of \( f(x) = 2x^2 - 3x + 4 \) at \( x = 1 \).

Carry out the same steps to find \( m_{\tan} : f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} : \)

\[
\begin{align*}
(1) & \quad f(1) = 2(1)^2 - 3(1) + 4 = 3 \\
(2) & \quad \frac{f(1 + h) - f(1)}{h} = \frac{2(1 + h)^2 - 3(1 + h) + 4 - 3}{h} \\
& \quad = \frac{2 + 4h + 2h^2 - 3 - 3h + 4 - 3}{h} = \frac{2h^2 + h}{h} = \frac{h(2h + 1)}{h} = 2h + 1 \\
(3) & \quad f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} (2h + 1) = 1
\end{align*}
\]

Example: Find the derivative of \( f(x) = \sqrt{3x - 2} \) at \( x = 2 \).

\[
\begin{align*}
(1) & \quad f(2) = \sqrt{3(2) - 2} = \sqrt{4} = 2 \\
(2) & \quad \frac{f(2 + h) - f(2)}{h} = \frac{\sqrt{3(2 + h) - 2} - 2}{h} = \frac{\sqrt{4 + 3h} - 2}{h} \\
& \quad = \frac{\sqrt{4 + 3h} - 2}{h} \left( \frac{\sqrt{4 + 3h} + 2}{\sqrt{4 + 3h} + 2} \right) = \frac{4 + 3h - 4}{h(\sqrt{4 + 3h} + 2)} \\
& \quad = \frac{3}{h(\sqrt{4 + 3h} + 2)} = \frac{3}{\sqrt{4 + 3h} + 2} \\
(3) & \quad f'(2) = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0} \frac{3}{\sqrt{4 + 3h} + 2} = \frac{3}{\sqrt{4} + 2} = \frac{3}{4}
\end{align*}
\]
2. Definition of the Derivative Function:
The derivative of a function \( f(x) \) is the function \( f'(x) \) defined as
\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]
provided the limit exists.
The process of computing a derivative is called differentiation.

Example: Without computing, find the derivative function \( f(x) \) for
\[a. \quad f(x) = c \quad b. \quad f(x) = mx + b\]
\[g(x) = \sqrt{2}, \quad g'(x) = 0. \quad f(x) = 2x + 1, \quad f'(x) = 2\]

Example: Find \( f'(x) \) if it exists where
\[a. \quad f(x) = 2x^2 - 3x + 4 \quad b. \quad f(x) = \frac{x+1}{2x-1} \text{ for } x \neq \frac{1}{2}\]
\[c. \quad f(x) = \sqrt{2x+1} \text{ for } x \geq -\frac{1}{2}\]

4. Alternative Derivative Notations:
The following are all alternative for the derivative function: let \( y = f(x) \)
\[
f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x)
\]
\(\frac{d}{dx}\) is also called a differential operator.

5. Nondifferentiable Points of a Function:
From the definition of \( f'(a) \), we know the function is not differentiable at \( x = a \) if the limit
\[
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]
does not exist.
Conditions with which \( f'(a) \) does not exist:
(i) \( f \) is not continuous at \( x = a \) (\( f(a) \) is not defined; \( \lim_{h \to 0} f(x) \) DNE or \( \lim_{h \to 0} f(x) \neq f(a) \));
(ii) \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \infty \); or
(iii) \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \text{DNE} \).
(\( f \) is always continuous at \( x = a \) if \( f'(a) \) exists.)

Example: Let \( f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \). Show graphically and algebraically that \( f \) is continuous at \( x = 0 \) but \( f(x) \) is not differentiable at \( x = 0 \).

Graphically, we see \( f \) is continuous at \( x = 0 \) and is not differentiable at \( x = 0 \).

Algebraically, \( f \) is continuous at \( x = 0 \) because
\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^2) = 0, \quad \lim_{x \to 0} f(x) = \lim_{x \to 0} (-x^2) = 0 = f(0).
\]

Now check \( \lim_{h \to 0^-} \frac{f(0 + h) - f(0)}{h} \) and \( \lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h} \):
\[
\frac{f(0 + h) - f(0)}{h} = \frac{f(h) - 0}{h} = \begin{cases} \frac{h^2}{h} = h & \text{if } h > 0 \\ \frac{-h}{h} = -1 & \text{if } h < 0 \end{cases}
\]

Hence, \( f \) is not differentiable at \( x = 0 \) since \( \lim_{h \to 0^-} \frac{f(0 + h) - f(0)}{h} \) DNE.

\[
\lim_{h \to 0^-} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-1}{h} = -1
\]
\[
\lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^+} \frac{0}{h} = 0.
\]