2.3 Computation of Derivatives: The Power Rule

1. The Power Rule:

Power function: $f(x) = x^n$, n a nonnegative integer

Example: $1, x, x^2, \ldots, x^{2008}$.

Graphically, we know the derivative of a power function **exists everywhere**. What is the derivative of a power function?

Power Rule: For any nonnegative integer n,

$$f(x) = x^n, \qquad f'(x) = nx^{n-1}$$

Generalized Power Rule:

For any real number r,

$$\frac{d}{dx}(x^r) = rx^{r-1}.$$

Proof:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$
$$= \lim_{h\to 0}\frac{x^n + nx^{n-1}h + \cdots + nxh^{n-1} + h^n - x^n}{h}$$
$$= \lim_{h\to 0}\frac{nx^{n-1}h + \cdots + nxh^{n-1} + h^n}{h}$$
$$= \lim_{h\to 0}\frac{h\left(nx^{n-1} + \cdots + nxh^{n-2} + h^{n-1}\right)}{h}$$
$$= \lim_{h\to 0}\left(nx^{n-1} + \cdots + nxh^{n-2} + h^{n-1}\right)$$
$$= nx^{n-1}.$$

Example: *Find* $f'(x)$ *if* i. $f(x) = x^{11}$;   ii. $f(x) = x^{2008}$

i. $f'(x) = 11x^{10}$ ii. $f'(x) = 2008x^{2007}$

Example: *Find* $f'(x)$ *if* i. $f(x) = x^{\pi}$ ii. $f(x) = \sqrt[3]{x^2}$

iii. $f(x) = \dfrac{1}{\sqrt{x^5}}$ iv. $f(x) = \dfrac{\sqrt{x^3}}{\sqrt[3]{x^2}}$ v. $f(x) = \sqrt{x}\,\sqrt[3]{x^5}$

i. $f(x) = x^{\pi}$, $f'(x) = \pi x^{\pi-1}$

ii. $f(x) = \sqrt[3]{x^2} = x^{2/3}$, $f'(x) = \frac{2}{3}x^{-1/3}$

iii. $f(x) = \frac{1}{\sqrt{x^5}} = x^{-5/2}$, $f'(x) = (-\frac{5}{2})x^{-7/2}$

iv. $f(x) = \dfrac{\sqrt{x^3}}{\sqrt[3]{x^2}} = \dfrac{x^{3/2}}{x^{2/3}} = x^{3/2-2/3} = x^{(9-4)/6} = x^{5/6}$, $f'(x) = \frac{5}{6}x^{-1/6}$

v. $f(x) = \sqrt{x}\,\sqrt[3]{x^5} = x^{1/2}x^{5/3} = x^{1/2+5/3} = x^{(3+10)/6} = x^{13/6}$

$f'(x) = \frac{13}{6}x^{5/6}$

2. Differentiation Rule for cf and $f \pm g$:

Let f and g be differentiable at x and c be a constant. Then

1. $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

2. $\frac{d}{dx}[cf(x)] = cf'(x)$

Example: *Find* $f'(x)$ *where*

i. $f(x) = 2x^{10} - \dfrac{3}{\sqrt{x}} + 4\sqrt[3]{x^5} + \pi$ ii. $f(x) = (x^2-1)(2x^3+x-1)$

iii. $f(x) = \dfrac{3x^2 - 4\sqrt{x} - 1}{x}$

i. $f(x) = 2x^{10} - 3x^{-1/2} + 4x^{5/3} + \pi$, $f'(x) = 20x^9 + \dfrac{3}{2}x^{-3/2} + \dfrac{20}{3}x^{2/3}$

ii. $f(x) = 2x^5 - 2x^3 + x^3 - x - x^2 + 1 = 2x^5 - x^3 - x^2 - x + 1$,

$f'(x) = 10x^4 - 3x^2 - 2x - 1$

iii. $f(x) = 3x - 4x^{-1/2} - x^{-1}$, $f'(x) = 3 + 2x^{-3/2} + x^{-2}$

Example: *Let* $f(x) = 3x^4 + 2x^3 + x^2 - 3x + 1$. *Find* f, f', f'', f''', $f^{(4)}$ *and* $f^{(n)}$ *for* $n \geq 5$.

$$f'(x) = 12x^3 + 6x^2 + 2x - 3, \quad f''(x) = 36x^2 + 12x + 2,$$
$$f'''(x) = 72x + 12, \quad f^{(4)}(x) = 72,$$
$$f^{(n)}(x) = 0 \text{ for } n \geq 5.$$

Example: Determine all value(s) of $f(x) = x^{1/4}$ at which f is **not differentiable**.

Answer: The domain of f is $D_f = [0, \infty)$.

$f'(x) = \frac{1}{4}x^{-3/4} = \frac{1}{4x^{3/4}}$.

The domain of $f'(x)$: $D_{f'} = (0, \infty)$.

Hence, f is not differentiable at $x = 0$.

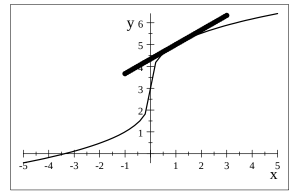
Example: *Find the equation of the tangent line to* $y = f(x)$ *where* $f(x) = 2\sqrt[3]{x} + 3$ *at the point where* $x = 1$.

Answer: $f(1) = 2(1) + 3 = 5$

$m_{\tan} = f'(1)$, $f(x) = 2x^{1/3} + 3$, $f'(x) = \frac{2}{3}x^{-2/3}$, $f'(1) = \frac{2}{3}$

The equation of the tangent line:

$$y - 5 = \frac{2}{3}(x - 1),$$
$$y = \frac{2}{3}x + \frac{13}{3}$$



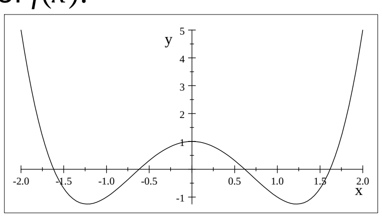
Example: *Determine the value of* x *for which the tangent line to* $y = f(x)$ *where* $f(x) = x^4 - 3x^2 + 1$ *is horizontal.*

Answer: The tangent line is horizontal if and only if it has 0 slope.

$f'(x) = 4x^3 - 6x = 2x(2x^2 - 3) = 0$, $x = 0$, $x = \pm\sqrt{\frac{3}{2}}$.

At $x = 0$, $x = \sqrt{\frac{3}{2}}$ and $x = -\sqrt{\frac{3}{2}}$, the tangent line has zero slope.

Verify with the graph of $f(x)$.



Example: *Let* $h(t) = -16t^2 + 10t + 2$ *in feet be the height of a moving object where* t *is in seconds.*

(1) Compute the **velocity** and **acceleration** of the object at time $t = 0$ and $t = 2$ seconds.

(2) Is the object going up or down?

(3) Is **the speed** of the object increasing or decreasing.

Answer:

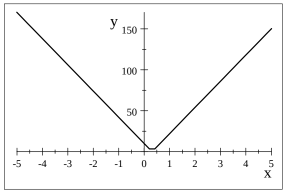
(1) $v(t) = h'(t) = -32t + 10$, feet/second,
$a(t) = h''(t) = -32$ feet^2/second.

$v(0) = 10$ feet/second, $a(0) = -32$ feet^2/second.

$v(2) = -54$ feet/second, $a(2) = -32$ feet^2/second.

(2) Since $v(0) > 0$, the object is going up at $t = 0$. Since $v(2) < 54$, the objects is going down at $t = 2$.

(3) Speed $s(t) = |v(t)| = |-32t + 10|$



which is decreasing before $t = \frac{10}{32} = \frac{5}{16}$ seconds and increasing after $t = \frac{5}{16}$ seconds. So, the speed is decreasing at $t = 0$ and is increasing at $t = 2$ seconds.