2.4 The Product and Quotient Rules

1. Product Rule:

Let f and g be differentiable at x. Then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)\ g(x) + f(x)\ g'(x).$$

Note that:

- $\frac{d}{dx}[f(x)g(x)] \neq f'(x)g'(x)$.
- $\frac{d}{dx}[f(x)g(x)h(x)] = \frac{d}{dx}[f(x)(g(x)h(x))]$
  $= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$
- For $n \geq 1$ integer, $\frac{d}{dx}[[f(x)]^n] = n[f(x)]^{n-1} f'(x)$.

$$\frac{d}{dx}[[f(x)]^n] = \underbrace{f'(x)\ \underbrace{f(x)\cdots f(x)}_{n-1 \text{ of } f(x)} + \cdots + \underbrace{f(x)\cdots f(x)}_{n-1 \text{ of } f(x)}\ f'(x)}_{n \text{ of } f(x)\cdots f(x)f'(x)} = n\,[f(x)]^{n-1} f'(x)$$

Proof. Let $h(x) = f(x)g(x)$. By definition,

$$h'(x) = \lim_{h\to 0} \frac{h(x+h) - h(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h} \right]$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} g(x+h) + \lim_{h\to 0} f(x) \frac{g(x+h) - g(x)}{h}$$

$$= f'(x)g(x) + f(x)g'(x)$$

2. Quotient Rule:

Let f and g be differentiable at x and $g(x) \neq 0$. Then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Note that:

- $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \neq \frac{f'(x)}{g'(x)}$$
- $$\frac{d}{dx}\left[[g(x)]^{-1}\right] = \frac{d}{dx}\left[\frac{1}{g(x)}\right] \overset{f(x)=0}{=} \frac{-g'(x)}{[g(x)]^2} = -g'(x)[g(x)]^2$$

  For $n \geq 1$ integer, $\frac{d}{dx}[[g(x)]^{-n}] = -ng'(x)[g(x)]^{-(n+1)}$.

  Proof:

  $$\frac{d}{dx}[[g(x)]^{-n}] = \frac{d}{dx}\left[\frac{1}{[g(x)]^n}\right] = \frac{-ng'(x)[g(x)]^{n-1}}{([g(x)]^n)^2}$$
  $$= -ng'(x)[g(x)]^{-(n+1)}$$

Example: *Find* $h'(x)$ *where*

a. $h(x) = (2x^2 - 1)\left(x^3 - \frac{1}{\sqrt{x}} + \frac{2}{x^2}\right)$

b. $h(x) = \frac{x^2 - 1}{3x^2 + 2}$

c. $h(x) = (\sqrt{x} + 1)\frac{x^2 - \sqrt{x}}{x + 2\sqrt{x}}$

d. $h(x) = \left(x + \frac{1}{x}\right)(2x^2 + x - 1)(\sqrt{x} + 1)$

e. $h(x) = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^{100}$

f. $h(x) = \frac{1}{(2x^2 - x + 1)^{2006}}$

a. $h(x) = (2x^2 - 1)\left(x^3 - \frac{1}{\sqrt{x}} + \frac{2}{x^2}\right) = f(x) = (2x^2 - 1)(x^3 - x^{-1/2} + 2x^{-2})$

$$h'(x) = 4x(x^3 - x^{-1/2} + 2x^{-2}) + (2x^2 - 1)\left(3x^2 + \frac{1}{2}x^{-3/2} - 4x^{-3}\right)$$

$$= 10x^4 - 3\sqrt{x} - 3x^2 - \frac{1}{2}x^{-\frac{3}{2}} + 4x^{-3}$$

b. $h(x) = \dfrac{x^2 - 1}{3x^2 + 2}$

$$h'(x) = \frac{2x(3x^2 + 2) - (x^2 - 1)(6x)}{(3x^2 + 2)^2} = \frac{10x}{(3x^2 + 2)^2}$$

c. $h(x) = (\sqrt{x} + 1)\dfrac{x^2 - \sqrt{x}}{x + 2\sqrt{x}} = (x^{1/2} + 1)\dfrac{x^2 - x^{1/2}}{x + 2x^{1/2}}$

$$h'(x) = \frac{1}{2}x^{-1/2}\frac{x^2 - x^{1/2}}{x + 2x^{1/2}}$$

$$+ (x^{1/2} + 1)\frac{(2x - \frac{1}{2}x^{-1/2})(x + 2x^{1/2}) - (x^{1/2} + 1)(x^2 - x^{1/2})}{(x + 2x^{1/2})^2}$$

d. $h(x) = \left(x + \frac{1}{x}\right)(2x^2 + x - 1)(\sqrt{x} + 1)$

$$h'(x) = (1 - x^{-2})(2x^2 + x - 1)(\sqrt{x} + 1) + \left(x + \frac{1}{x}\right)(4x + 1)(\sqrt{x} + 1)$$

$$+ \left(x + \frac{1}{x}\right)(2x^2 + x - 1)\left(\frac{1}{2}x^{-1/2} + 1\right)$$

e. $h(x) = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^{100}$

$$h'(x) = 100\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^{99} \frac{d}{dx}[x^{1/2} + x^{-1/2}]$$

$$= 100\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^{99} \left(\frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}\right)$$

f. $h(x) = \frac{1}{(2x^2 - x + 1)^{2006}} = (2x^2 - x + 1)^{-2006}$

$$h'(x) = -2006(4x - 1)^{-2007}$$

Example: *Suppose that* $f(x)$ *and* $g(x)$ *are differentiable and we know*

$$f(0) = -1,\ f(1) = -2,\ f'(0) = -1,\ f'(1) = 3$$

$$g(0) = 3,\ g(1) = 1,\ g'(0) = -2,\ g'(1) = -3.$$

Find the equation of the tangent line to the curve $y = h(x)$ *at* $x = a$.

i. $h(x) = f(x)g(x),\ x = 0$ ii. $h(x) = \dfrac{x^2 f(x)}{g(x)},\ x = 1$

i. $m_{\tan} = h'(0) = f'(0)g(0) + f(0)g'(0) = (-1)(3) + (-1)(-2) = -1$

Equation of the tangent line: $h(0) = f(0)g(0) = (-1)(3) = -3$

$$y - h(0) = h'(0)(x - 0),\quad y = (-1)x + (-3) = -x - 3$$

ii. $m_{\tan} = h'(1),\ h'(x) = \dfrac{[2xf(x) + x^2 f'(x)]g(x) - x^2 f(x)g'(x)}{[g(x)]^2}$

$$h'(1) = \frac{\left[2(1)f(1) + (1)^2 f'(1)\right]g(1) - (1)^2 f(1)g'(1)}{[g(1)]^2} = -3$$

tangent line: $h(1) = \dfrac{(1)^2 f(1)}{g(1)} = -2,\ y + 2 = (-3)(x - 1),\ y = -3x + 1.$

Example: *A baseball with mass 0.15 kg and speed 45m/s is struck by a baseball bat of mass* m *and speed* 40 *m/s (in the opposite direction of the ball's motion). After the collision, the ball has initial speed*

$$u(m) = \frac{82.5m - 6.75}{m + 0.15}\ m/s.$$

Find $\lim_{m\to\infty} u(m)$. *Show that* $u'(m) > 0$, *and compare* $u'(1)$ *and* $u'(1.2)$.

$$\lim_{m\to\infty} u(m) = \lim_{m\to\infty} \frac{82.5m - 6.75}{m + 0.15} = \lim_{m\to\infty} \frac{82.5m}{m} = 82.5\ m/s.$$

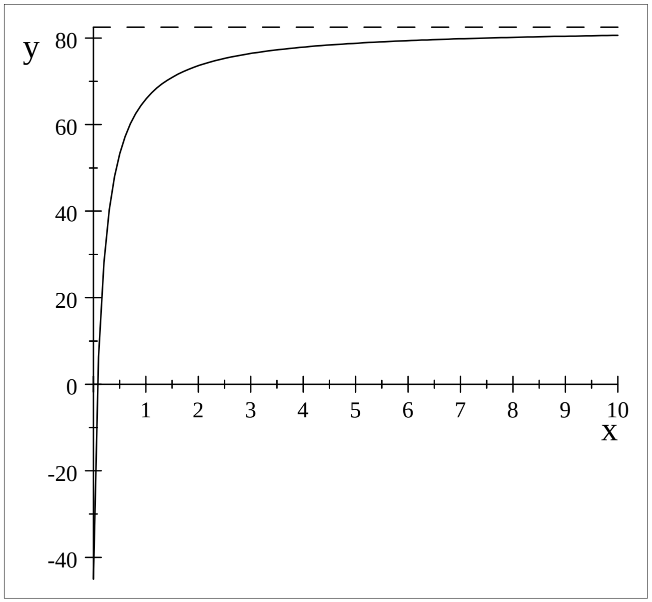
$$u'(m) \overset{\text{QR}}{=} \frac{(82.5)(m + 0.15) - (82.5m - 6.75)(1)}{(m + 0.15)^2}$$

$$= \frac{(82.5)(0.15) + 6.75}{(m + 0.15)^2} = \frac{19.125}{(m + 0.15)^2} > 0$$

This means the initial speed is always increasing with respect to the mass of a baseball bat. Let us check the values of $u'(m)$ at $m = 1$ and $m = 1.2$:

$$u'(1) = \frac{19.125}{(1.15)^2} = 14.4612476 \text{ m/s/kg}$$

$$u'(1.2) = \frac{19.125}{(1.35)^2} = 10.4938272 \text{ m/s/kg.}$$



$y = u(m)$

$u(m)$ is increasing but its increasing rate is decreasing ($u''(m)$ should be negative). Though the initial speed is increasing, it will be always less than 82.5 m/s (Since $\lim_{m\to\infty} u(m) = 82.5$ m/s) no matter how large the mass of bat is.