2.6 Derivatives of Trigonometric Functions

Derivatives of $\sin(x)$ and $\cos(x)$:

Facts: (1) $\lim_{h\to 0} \frac{\sin h}{h} = 1$ (2) $\lim_{h\to 0} \frac{1-\cos h}{h} = 0$

Proof (of (2)):

$$\lim_{h\to 0} \frac{1-\cos h}{h} = \lim_{h\to 0} \frac{2\sin^2(\frac{h}{2})}{h} = \lim_{h\to 0} \frac{\sin(\frac{h}{2})}{\frac{h}{2}} \sin\left(\frac{h}{2}\right) = (1)(0) = 0$$

Derivatives of $\sin x$ and $\cos x$:

(1) $\frac{d}{dx}[\sin x] = \cos x$ (2) $\frac{d}{dx}[\cos x] = -\sin x$

Derivatives of $\tan x$, $\cot x$, $\sec x$, **and** $\csc x$:

(3) $\frac{d}{dx}[\tan x] = \sec^2 x$ (4) $\frac{d}{dx}[\cot x] = -\csc^2 x$

(5) $\frac{d}{dx}[\sec x] = \tan x \sec x$ (6) $\frac{d}{dx}[\csc x] = -\cot x \csc x$

Recall: $\sin(a \pm b) = \sin(a)\cos(b) \pm \sin(b)\cos(a)$

$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$

Derivation of $\frac{d}{dx}[\sin x]$:

$$\begin{aligned}
\frac{d}{dx}[\sin x] &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1) + \sin(h)\cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)}{h} + \lim_{h\to 0} \frac{\sin(h)\cos(x)}{h} \\
&= \sin(x)(0) + (1)\cos(x))
\end{aligned}$$

Derivation of $\frac{d}{dx}[\cos x]$: HW

Recall: $\sin^2 x + \cos^2 x = 1$

$\sec^2 x - \tan^2 x = 1$

Derivations of $\frac{d}{dx}[\tan x]$ and $\frac{d}{dx}[\sec x]$.

$$\begin{aligned}
\frac{d}{dx}[\tan x] &= \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] \overset{\text{Q.R.}}{=} \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x \cos x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \\
\frac{d}{dx}[\sec x] &= \frac{d}{dx}\left[\frac{1}{\cos x}\right] \overset{\text{Q.R.}}{=} \frac{0 - (1)(-\sin x)}{\cos^2 x} \\
&= \frac{\sin x}{\cos x}\left(\frac{1}{\cos x}\right) = \tan x \sec x
\end{aligned}$$

Derivations of $\frac{d}{dx}[\cot x]$ and $\frac{d}{dx}[\csc x]$: HW

Example: Let $f(x) = \sin x$ and $g(x) = \cos x$. Find $f^{(5)}(x)$, $f^{(2006)}(x)$, $g^{(5)}(x)$ and $g^{(2006)}(x)$.

n	0	1	2	3	4
$f^{(n)}(x)$	$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$
$g^{(n)}(x)$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\cos x$

$5 = 4 + 1$,

$$f^{(5)}(x) = f'(x) = \cos x,\ g^{(5)}(x) = g'(x) = -\sin x$$

$2006 = 4(501) + 2$,

$$f^{(2006)}(x) = f''(x) = -\sin x,\ g^{(2006)}(x) = g''(x) = -\cos x$$

Example: *Compute $f'(x)$ where*

a. $f(x) = \dfrac{x^2}{\sin^2 x}$ b. $f(x) = 2\sin(\pi x)\cos(3x)$

c. $f(x) = \sqrt{\cos^2 x + x^2}$ d. $f(x) = \sec^2 x - \sec(x^2)$

e. $f(x) = \sqrt{x}\tan(\sqrt{x})\cot(3x)$ f. $f(x) = 3\cos(\sec(x^2))$

a. $f(x) = \dfrac{x^2}{\sin^2 x} = x^2\csc^2 x$

$$f'(x) = \frac{2x\sin^2 x - x^2(2\sin x\cos x)}{\sin^4 x} = \frac{2x\sin x(\sin x - x\cos x)}{\sin^4 x}$$

$$f'(x) = 2x\csc^2 x + x^2(2\csc x(-\csc x\cot x)) = 2x\csc^2 x(1 - x\cot x)$$

b. $f(x) = 2\sin(\pi x)\cos(3x)$

$$f'(x) = 2(\cos(\pi x)(\pi)\cos(3x) + \sin(\pi x)(-\sin(3x)(3))) = 2(\pi\cos(\pi x)\cos(3x) - 3\sin(\pi x)\sin(3x))$$

c. $f(x) = \sqrt{\cos^2 x + x^2} = (\cos^2 x + x^2)^{1/2}$

$$f'(x) = \frac{1}{2}(\cos^2 x + x^2)(2\cos x(-\sin x) + 2x) = (\cos^2 x + x^2)(x - \cos x\sin x)$$

d. $f(x) = \sec^2 x - \sec(x^2)$

$$f'(x) = 2\sec x(\sec x\tan x) - \sec(x^2)\tan(x^2)(2x) = 2[\sec^2 x\tan x - \sec(x^2)\tan(x^2)]$$

e. $f(x) = \sqrt{x}\tan(\sqrt{x})\cot(3x)$

$$f'(x) = \frac{1}{2}x^{-1/2}\tan(\sqrt{x})\cot(3x) + \sqrt{x}\sec^2(\sqrt{x})\left(\frac{1}{2}x^{-1/2}\right)\cot(3x) + \sqrt{x}\tan(\sqrt{x})(-3\csc^2(3x))$$

$$= \frac{1}{2}x^{-1/2}\tan(\sqrt{x})\cot(3x) + \frac{1}{2}\sec^2(\sqrt{x})\cot(3x) - 3\sqrt{x}\tan(\sqrt{x})\csc^2(3x)$$

f. $f(x) = 3\cos(\sec(x^2))$

$$f'(x) = 3[-\sin(\sec(x^2))(\sec(x^2)\tan(x^2))(2x)] = -6\sin(\sec(x^2))(\sec(x^2)\tan(x^2))$$

Example: *Find the equation of the tangent line to the curve* $y = x^2\cos x$ *at* $a = \frac{\pi}{3}$.

Equation of the tangent line:

$$y - f\left(\frac{\pi}{3}\right) = f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right)$$

$$f\left(\frac{\pi}{3}\right) = \left(\frac{\pi}{3}\right)^2\cos\left(\frac{\pi}{3}\right) = \left(\frac{\pi}{3}\right)^2\left(\frac{1}{2}\right) = \frac{\pi^2}{18}$$

$$f'(x) = 2x\cos x + x^2(-\sin x) = 2x\cos x - x^2\sin x$$

$$f'\left(\frac{\pi}{3}\right) = (2)\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) - \left(\frac{\pi}{3}\right)^2\sin\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \left(\frac{\pi}{3}\right)^2\frac{\sqrt{3}}{2}$$

tangent line: $y - \dfrac{\pi^2}{18} = \left(\dfrac{\pi}{3} - \left(\dfrac{\pi}{3}\right)^2\dfrac{\sqrt{3}}{2}\right)\left(x - \dfrac{\pi}{3}\right)$

Example: *Find all values of x at which the tangent line to the curve $y = \frac{1}{2}x + \sin x$ is horizontal.*

Slope function: $m = f'(x) = \frac{d}{dx}[\frac{1}{2}x + \sin x] = \frac{1}{2} + \cos(x)$.

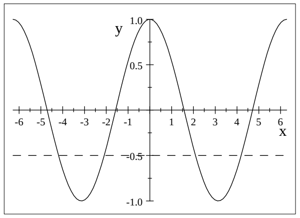
Set $f'(x) = 0$ and solve for x : $\frac{1}{2} + \cos(x) = 0$,

$\cos(x) = -\frac{1}{2}$, x is in the 2nd and 3rd quadrant.

For $0 \le x < 2\pi$, $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$, $x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

For $-\infty < x < \infty$, $x = \frac{2\pi}{3} \pm 2n\pi$, $x = \frac{4\pi}{3} \pm 2n\pi$, $n = 0, 1, \ldots, n$

Verify the results graphically:



Example: *Let $s(t) = 4\cos(2t)$ in meters be a position function of a moving object at the time t seconds. Find the velocity of the object at time $t = 0$ and $\frac{\pi}{4}$. For what time is the velocity 0? What is the location of the object when its velocity is 0? When does the object change directions?*

(1) $v(t) = s'(t) = 4(-\sin(2t)(2)) = -8\sin(2t)$,

$v(0) = 0$ m/s

$v\left(\frac{\pi}{4}\right) = -8\sin\left(2\left(\frac{\pi}{4}\right)\right) = -8\sin\left(\frac{\pi}{2}\right) = -8$ m/s

(2) $v(t) = 0$, $-8\sin(2t) = 0$

$2t = \pm n\pi = n\pi$, $t = \frac{n\pi}{2}$, $n = 0, 1, 2, \ldots$.

(3) When $t = \frac{n\pi}{2}$, $n = 0, 1, 2, \ldots$,

$s(t) = 4\cos(2t) = 4\cos\left(2\left(\frac{n\pi}{2}\right)\right) = 4\cos(n\pi) = \pm 4$

(4) The object changes directions whenever its velocity is 0 :
$t = \frac{n\pi}{2}$, $n = 0, 1, 2, \ldots$.