2.7 Derivatives of Exponential and Logarithmic Functions

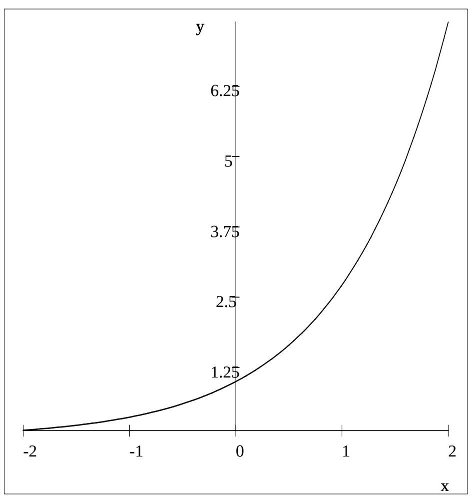
1. Derivatives of the Exponential Functions

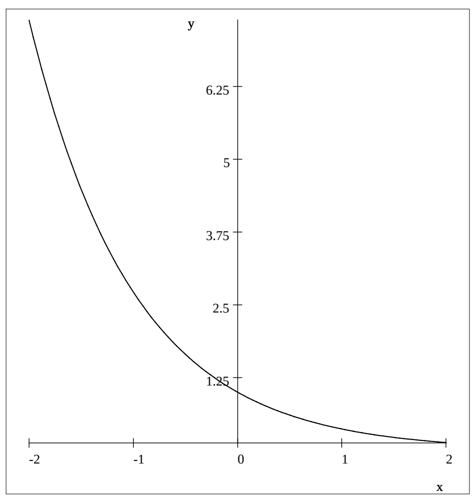
Let $a > 0$ and $a \neq 1$, and $f(x) = a^x$. Then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} \frac{a^x a^h - a^x}{h}$$

$$= \lim_{h\to 0} \frac{a^x(a^h - 1)}{h} = a^x \left[\lim_{h\to 0} \frac{a^h - 1}{h} \right]$$

Questions: Does $\lim_{h\to 0} \frac{a^h - 1}{h}$ exist? If so, what is the limit?





The graph of $y = a^x$ shows that a^x is differentiable everywhere. So the limit should exist.

Observe that $\lim_{h\to 0} \dfrac{a^h - 1}{h} = f\,'(0)$ and

$$\frac{d}{dx}(a^x) = \left(f\,'(0)\right) a^x = \text{(a constant) } a^x.$$

Evaluate numerically $\lim_{h\to 0} \dfrac{a^h - 1}{h}$: $a = 2$

h	$\dfrac{2^h - 1}{h}$	h	$\dfrac{2^h - 1}{h}$
0.0001	0.6931712038	-0.0001	0.6931231585
0.00001	0.6931712038	-0.00001	0.6931447783
0.000001	0.69314742	-0.000001	0.69314694
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
0^+	0.6931465	0^-	0.6931465

$$\lim_{h\to 0} \frac{2^h - 1}{h} \approx 0.6931465 (\approx \ln(2))$$

$a = 3:$

h	$\dfrac{3^h - 1}{h}$	h	$\dfrac{3^h - 1}{h}$
0.0001	1.098672638	-0.0001	1.098551943
0.00001	1.098618323	-0.00001	1.098606254
0.000001	1.09861289	-0.000001	1.098611685
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
0^+	1.098615	0^-	1.098615

$$\lim_{h \to 0} \frac{3^h - 1}{h} \approx 1.098615 (\approx \ln(3))$$

Fact (it will be proved in Cal. II):

$$\lim_{h\to 0}\frac{a^h-1}{h} = \ln a \qquad \left(\lim_{h\to 0}\frac{e^h-1}{h} = \ln e = 1\right).$$

Therefore:

$$\frac{d}{dx}[a^x] = a^x \ln a \qquad \left(\frac{d}{dx}[e^x] = e^x\right)$$

Combine it with the Chain Rule:

$$\frac{d}{dx}[a^{g(x)}] = a^{g(x)} \ln a \, g'(x) \qquad \left(\frac{d}{dx}[e^{g(x)}] = e^{g(x)}\right)$$

Example: Find $f'(x)$ where

a. $f(x) = e^x + 4^x + x^4 + \pi^x + x^\pi$

b. $f(x) = \sqrt[3]{x}\ \sqrt{e^x}$

c. $f(x) = \dfrac{3^x}{e^x}$

d. $f(x) = \left(\dfrac{1}{2}\right)^x \sin(e^{2x})$

a. $f(x) = \frac{1}{2}4^x - e^x + \pi^x$

$$f'(x) = \frac{1}{2}(4^x \ln(4)) - e^x - \pi^x \ln(\pi)$$

b. $f(x) = x^{1/3}e^{x/2}$

$$f'(x) = \frac{1}{3}x^{-2/3}e^{x/2} + x^{1/3}\left(\frac{1}{2}e^{x/2}\right) = \frac{1}{3}x^{-2/3}e^{x/2} + \frac{1}{2}x^{1/3}e^{x/2}$$

c. $f(x) = \frac{3^x}{e^x}$

$$f'(x) = \frac{(3^x \ln 3)(e^x) - 3^x(e^x)}{(e^x)^2} = \frac{e^x 3^x(\ln 3 - 1)}{e^{2x}} = \frac{3^x(\ln 3 - 1)}{e^x}$$

d. $f(x) = \left(\frac{1}{2}\right)^x \sin(e^{2x})$

$$f'(x) = (2^{-x}(-1))\sin(e^{2x}) + 2^{-x}\cos(e^{2x})(e^{2x})(2)$$

$$= -2^{-x}\sin(e^{2x}) + 2^{x+1}(e^{2x})\cos(e^{2x})$$

2. **Derivatives of Logarithmic Functions**

$$\frac{d}{dx}[\log_a x] = ? \qquad \frac{d}{dx}[\ln x] = ?$$

Let $f(x) = e^x$ and $g(x) = \ln x,\ x > 0.$ Then $e^{\ln x} = x,$ that is

$$\frac{d}{dx}[e^{\ln x}] = \frac{d}{dx}[x] = 1, \quad \frac{d}{dx}[e^{\ln x}] \overset{\text{Chain Rule}}{=} e^{\ln x}\frac{d}{dx}[\ln x] = 1$$

$$\frac{d}{dx}[\ln x] = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

Can we derive $\frac{d}{dx}[\log_a x]$ from $\frac{d}{dx}[\ln x] = \frac{1}{x}$? Yes,

$$\frac{d}{dx}[\log_a x] = \frac{d}{dx}\left[\frac{\ln x}{\ln a}\right] = \frac{1}{\ln a}\left[\frac{d}{dx}[\ln x]\right] = \frac{1}{(\ln a)x}, \quad x > 0$$

Combine it with the Chain Rule:

$$\frac{d}{dx}[\ln(g(x))] = \frac{1}{g(x)}g'(x) = \frac{g'(x)}{g(x)} \qquad \left(\frac{d}{dx}[\log_a(g(x))] = \frac{g'(x)}{(\ln a)g(x)}\right)$$

Example: Find $f'(x)$ where

a. $f(x) = \ln(4x) + \ln(x^2 + e^x)$

b. $f(x) = x^2 \ln\sqrt{e^x + x}$

c. $f(x) = \ln(\sin^3(x^2))$

d. $f(x) = \sqrt{\ln(x^2 + 1)}$

a. $f(x) = \ln(4x) + \ln(x^2 + e^x) = \ln 4 + \ln x + \ln(x^2 + e^x)$

$$f'(x) = 0 + \frac{1}{x} + \frac{2x + e^x}{x^2 + e^x} = \frac{1}{x} + \frac{2x + e^x}{x^2 + e^x}$$

b. $f(x) = x^2 \ln\sqrt{e^x + x} = x^2(\frac{1}{2}\ln(e^x + x)) = \frac{1}{2}x^2 \ln(e^x + x)$

$$f'(x) = \frac{1}{2}\left(2x\ln(e^x + x) + x^2\frac{e^x + 1}{e^x + x}\right)$$

c. $f(x) = \ln(\sin^3(x^2)) = \ln\left((\sin(x^2))^3\right) = 3\ln(\sin(x^2))$

$$f'(x) = 3\left(\frac{\cos(x^2)(2x)}{\sin(x^2)}\right) = 6x\cot(x^2)$$

d. $f(x) = \sqrt{\ln(x^2 + 1)} = [\ln(x^2 + 1)]^{1/2}$

$$f'(x) = \frac{1}{2}[\ln(x^2 + 1)]^{-1/2}\left[\frac{2x}{x^2 + 1}\right] = \left(\frac{x}{x^2 + 1}\right)[\ln(x^2 + 1)]^{-1/2}$$

3. $\frac{d}{dx}\left[[g(x)]^{h(x)}\right] = ?$ Note that $g(x) = e^{\ln g(x)}$.

$$\frac{d}{dx}\left[[g(x)]^{h(x)}\right] = \frac{d}{dx}\left[[e^{\ln g(x)}]^{h(x)}\right] = \frac{d}{dx}[e^{h(x)\ln g(x)}]$$

$$\overset{\text{Chain Rule}}{=} e^{h(x)\ln g(x)} \frac{d}{dx}[h(x)\ln g(x)]$$

$$\overset{\text{Product Rule+Chain Rule}}{=} e^{h(x)\ln g(x)}\left[h'(x)\ln g(x) + h(x)\frac{1}{g(x)}g'(x)\right]$$

$$= [g(x)]^{h(x)}\left[h'(x)\ln g(x) + h(x)\frac{g'(x)}{g(x)}\right]$$

Hence, $$\frac{d}{dx}\left[[g(x)]^{h(x)}\right] = [g(x)]^{h(x)}\frac{d}{dx}[h(x)\ln g(x)]$$

Example: Compute the following derivatives.

a. $\frac{d}{dx}[x^x]$ b. $\frac{d}{dx}[x^{\sin x}]$ c. $\frac{d}{dx}[(\ln x)^x]$ d. $\frac{d}{dx}\left[x^{x^2-x}\right]$

a. $$\frac{d}{dx}[x^x] \underset{h(x)=x}{\overset{g(x)=x}{=}} x^x \frac{d}{dx}[x\ln(x)] = x^x\left[\ln x + x\left(\frac{1}{x}\right)\right] = x^x(\ln x + 1)$$

b. $$\frac{d}{dx}[x^{\sin x}] \underset{h(x)=\sin x}{\overset{g(x)=x}{=}} = x^{\sin x}\frac{d}{dx}[\sin(x)\ln x] = x^{\sin x}\left[\cos(x)\ln(x) + \sin(x)\left(\frac{1}{x}\right)\right]$$

c. $$\frac{d}{dx}[(\ln x)^x] \underset{h(x)=x}{\overset{g(x)=\ln x}{=}} = (\ln x)^x \frac{d}{dx}[x\ln(\ln(x))]$$

$$= (\ln x)^x\left[(1)\ln(\ln x) + x\left(\frac{\frac{1}{x}}{\ln x}\right)\right] = (\ln x)^x\left[\ln(\ln x) + \frac{1}{\ln x}\right]$$

d. $$\frac{d}{dx}\left[x^{x^2-x}\right] \underset{h(x)=x^2-x}{\overset{g(x)=x}{=}} x^{x^2-x}\frac{d}{dx}[(x^2-x)\ln(x)]$$

$$= x^{x^2-x}\left[(2x-1)\ln x + (x^2-x)\frac{1}{x}\right]$$

4. $\frac{d}{dx}[f^{-1}(x)] = ?$ Note that $f(f^{-1}(x)) = x$.

$$\frac{d}{dx}[f(f^{-1}(x))] = f'(f^{-1}(x)) \cdot \frac{d}{dx}[f^{-1}(x)] = \frac{d}{dx}[x] = 1$$

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}.$$

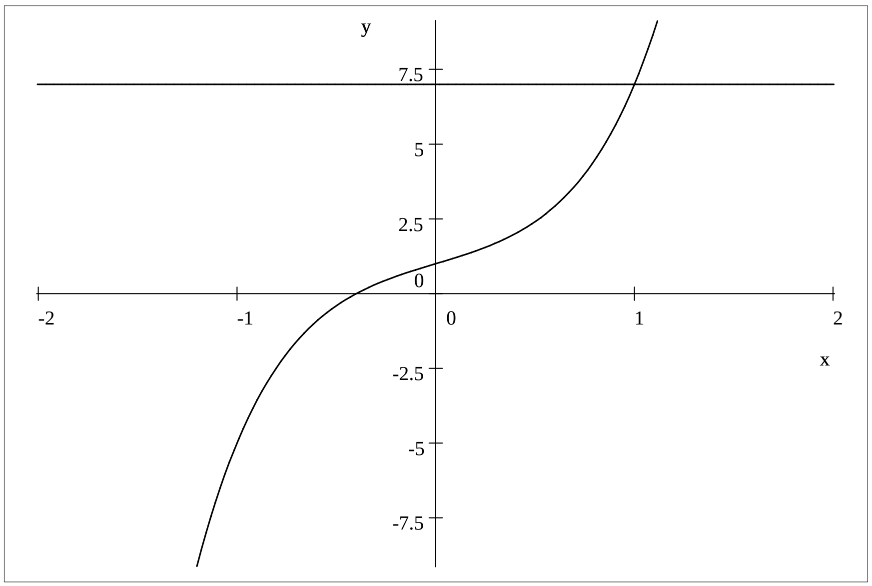
Example: Let $f(x) = x^5 + 3x^3 + 2x + 1$. $f(x)$ has an inverse function $g(x)$. Find $g'(7)$.

We know

$$g'(x) = \frac{1}{f'(g(x))}, \text{ and } g'(7) = \frac{1}{f'(g(7))}$$

(1) Compute $f'(x) = 5x^4 + 9x^2 + 2$.

(2) Compute $g(7) = f^{-1}(7)$: set $f(x) = 7$ and solve for x :



Graphically, $x = 1$.

$$f^{-1}(7) = g(7) = 1$$

$$g'(x) = \frac{1}{f'(g(7))} = \frac{1}{f'(1)}$$

$$= \frac{1}{16}$$

Example: *Find the equation of the tangent line to the curve* $y = (x^2 - 1)e^x$ *at the point where* $x = 0$. *Find all values of* x *at which the tangent line to the curve is horizontal.*

Let $f(x) = (x^2 - 1)e^x$. $\quad f(0) = (0 - 1)e^0 = -1$

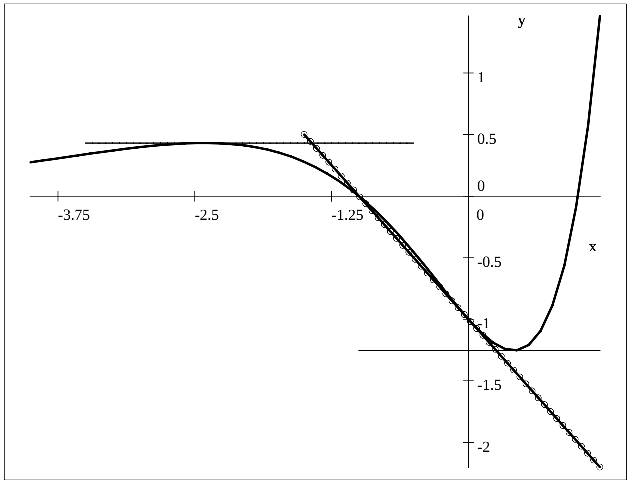
$f'(x) = 2xe^x + (x^2 - 1)e^x, \quad f'(0) = 0 - e^0 = -1$

The equation of the tangent line:

$y - f(0) = f'(0)(x - 0),$

$y + 1 = (-1)x, \quad y = -x - 1.$

Verify the result with the graphs:



$- y = (x^2 - 1)e^x$

Set $f'(x) = 0$ and solve for x :

$$2xe^x + (x^2 - 1)e^x = (x^2 + 2x - 1)e^x = 0 \Leftrightarrow x^2 + 2x - 1 = 0$$

$$x = \frac{-2 \pm \sqrt{4 - 4(-1)}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$$

So, at the points where $x = -1 + \sqrt{2}$ and $x = -1 - \sqrt{2}$ the tangent line to the curve is horizontal. The result is verified by the graph.

<u>**Example**</u>: Find the velocity and acceleration of a moving object at $t = 1$ minute if we know the position function of this object is $s(t) = \dfrac{\ln t}{\sqrt{t}}$ in feet. Determine if the object is increasing or decreasing at $t = 1$.

(1) $v(t) = s'(t) = \dfrac{d}{dt}\left[\dfrac{\ln t}{t^{1/2}}\right] = \dfrac{\frac{1}{t}t^{1/2} - \ln t(\frac{1}{2}t^{-1/2})}{(t^{1/2})^2}\left(\dfrac{t^{1/2}}{t^{1/2}}\right)$

(2) $v(1) = \dfrac{\frac{1}{1}(1)^{1/2} - \ln(1)\left(\frac{1}{2}(1)^{-1/2}\right)}{1} = 1$ ft/sec.

Since $v(1) > 0,$ the object is increasing.

(3) Simplify $v(t) = \dfrac{\frac{1}{t}t^{1/2} - \ln t(\frac{1}{2}t^{-1/2})}{t}\left(\dfrac{2t^{1/2}}{2t^{1/2}}\right) = \dfrac{1}{2}\,\dfrac{2 - \ln t}{t^{3/2}}$

$a(t) = v'(t) = \dfrac{1}{2}\,\dfrac{\left(0 - \frac{1}{t}\right)(t^{3/2}) - (2 - \ln t)(\frac{3}{2}t^{1/2})}{(t^{3/2})^2}$

$$= \frac{1}{2}\,\frac{-t^{1/2} - (2 - \ln t)(\frac{3}{2}t^{1/2})}{t^3}$$

$$a(1) = \frac{1}{2}\,\frac{-(1)^{1/2} - (2 - \ln 1)\left(\frac{3}{2}(1)^{1/2}\right)}{(1)^3} = \frac{-1 - 3}{2} = -2 \text{ ft/(sec sec)}$$

Example: If the value of an investment starts with $\$1000$ and doubles every 3 years, what is its value after t years? Find the instantaneous percentage rate of change of the worth.

Let $v(t) = 1000a^t$. Find the constant a so that $v(3) = 2000$.

$$v(3) = 1000\ a^3 = 2000, \quad a^3 = 2, \quad a = \sqrt[3]{2}, \ v(t) = 1000\left(\sqrt[3]{2}\right)^t.$$

The **instantaneous rate of change** of the investment:

$$v'(t) = 1000\left(\sqrt[3]{2}\right)^t \ln(2^{1/3}) = 1000\left(\frac{1}{3}\ln 2\right)\left(\sqrt[3]{2}\right)^t$$

The **relative rate of change** of the investment:

$$\frac{v'(t)}{v(t)} = \frac{1000(\frac{1}{3}\ln 2)\left(\sqrt[3]{2}\right)^t}{1000\left(\sqrt[3]{2}\right)^t} = \frac{1}{3}\ln 2 \approx 0.2310$$

The **instantaneous percentage rate of change**:

$$\frac{v'(t)}{v(t)} \times 100\% \approx 0.2310 \times 100\% = 23.10\ \%$$

Example: The motion of a spring is described by $s(t) = e^{-2t}\cos(3t)$.

(1) Compute the velocity at time t.

(2) Graph the velocity function.

(3) When is the velocity zero?

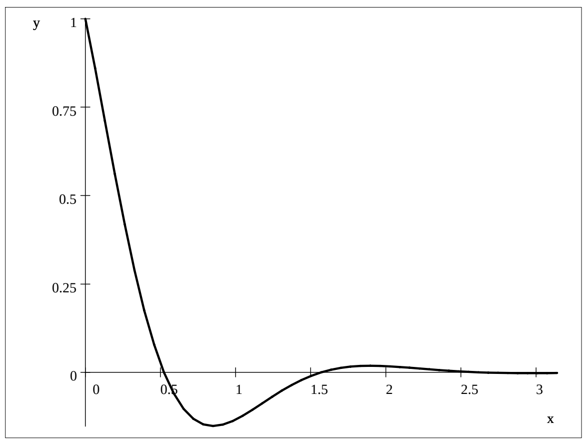
(4) What is the position of the spring when the velocity is zero?

(1)

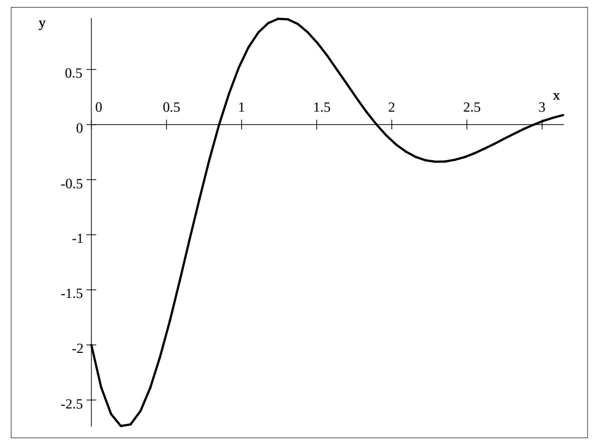
$$v(t) = s'(t) = -2e^{-2t}\cos(3t) + e^{-2t}(-3\sin(3t)) = e^{-t}(-2\cos(3t) - 3\sin(3t))$$

(2)

t	$v(t)$
0	1
$\frac{\pi}{6}$	0
$\frac{\pi}{3}$	$-e^{-2\pi/3}$
$\frac{\pi}{2}$	0



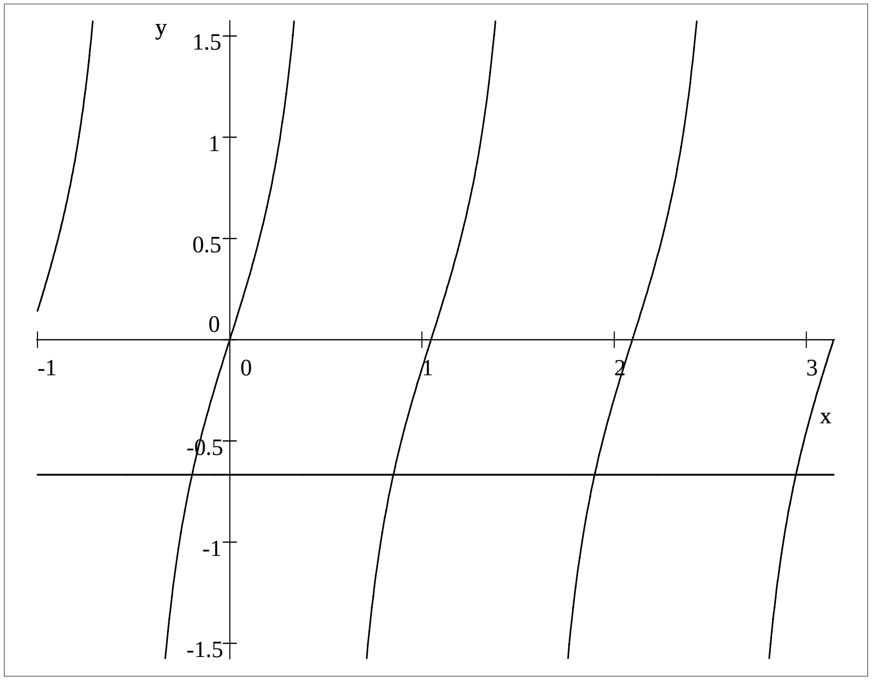
$s(t)$



$v(t)$

(3) $v(t) = 0 \Leftrightarrow e^{-t}(-2\cos(3t) - 3\sin(3t)) = 0$

$\Leftrightarrow (-2\cos(3t) - 3\sin(3t)) = 0.$



Period of $\tan(3t)$ is $\frac{\pi}{3}$.

$$-2\cos(3t) = 3\sin(3t)$$

$$\tan(3t) = -\frac{2}{3}$$

$$3t = \tan^{-1}(-\frac{2}{3}) = -0.588$$

$$t = \frac{1}{3}(-0.588) = -0.196$$

all solutions: $t = -0.196 + n(\frac{\pi}{3})$

$n = 1, 2, 3, \ldots.$

(4) Locations of the spring when its velocity is 0:

When $v(t) = 0$, $\cos(3t)$ is either 1 or -1. When $n = 1$, $\cos(3t) = -1$.

Hence,

t	$s(t)$
$-0.196+\frac{\pi}{3}=0.851$	$-e^{-2(0.851)}=0.1823$
$-0.196+2(\frac{\pi}{3})=1.898$	$e^{-2(1.898)}=0.0225$
$-0.196+3(\frac{\pi}{3})=2.946$	$-e^{-2(2.946)}=0.002761$
$-0.196+4(\frac{\pi}{3})=3.9928$	$e^{-2(3.9928)}=0.00034$
$\vdots$	$\vdots$