2.8 Part II - Derivatives of Inverse of Trigonometric

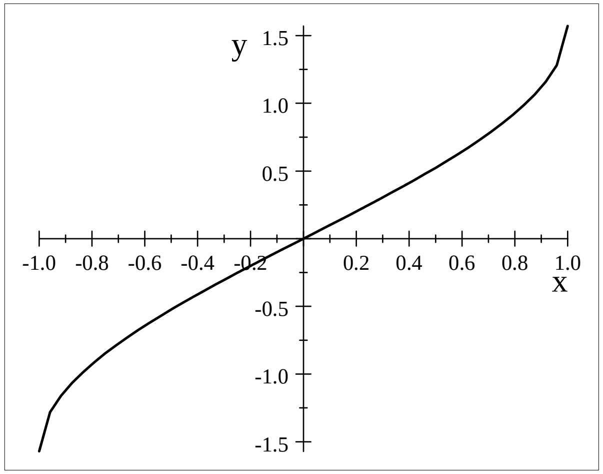
1. Review Inverse Trigonometric Functions:

Inverse Sine:

(1) $y = \sin^{-1}(x) = \arcsin(x)$

Domain $D_{\sin^{-1}x} = [-1,\ 1]$

Range $R_{\sin^{-1}x} = \left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$



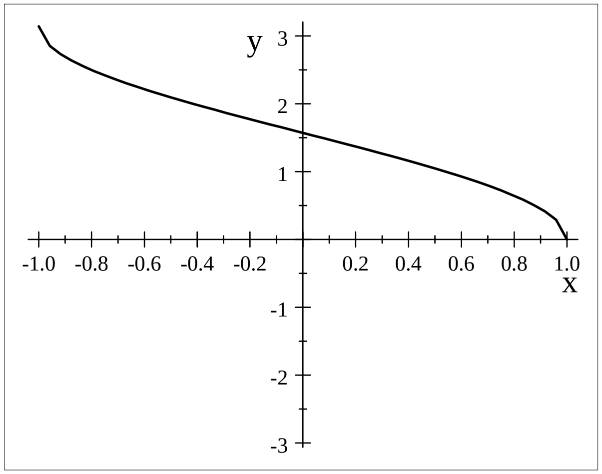
$y = \sin^{-1}x,\ x$ in

Inverse Cosine

(2) $y = \cos^{-1}(x) = \arccos(x)$

Domain: $D_{\cos^{-1}x} = [-1,\ 1]$

Range: $R_{\cos^{-1}x} = [0,\ \pi]$



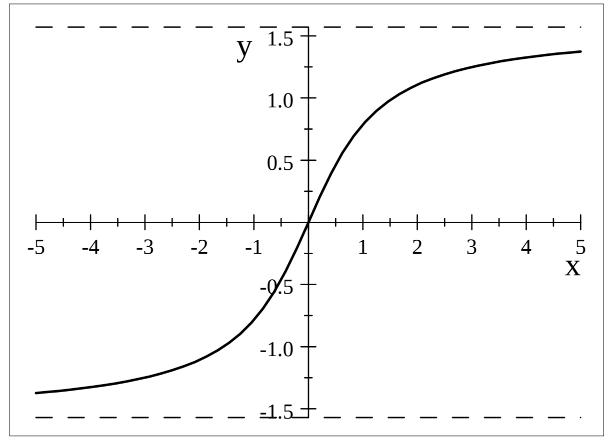
$y = \cos^{-1}x,\ x$ in

(3) **Inverse tangent**

$y = \tan^{-1}x$

Domain: $D_{\tan^{-1}x} = (-\infty, \infty)$

Range: $R_{\tan^{-1}x} = \left(-\dfrac{\pi}{2},\ \dfrac{\pi}{2}\right)$



$y = \tan^{-1}x,\ x$ in

(4) **Inverse Secant**, **Inverse Cosecant and Inverse Cotangent**:

Relation of $\sec^{-1}x$ and $\cos^{-1}x$: let $y = \sec^{-1}x$, then $\sec y = x$

$$\frac{1}{\cos y} = x, \quad \cos y = \frac{1}{x}, \quad y = \cos^{-1}\left(\frac{1}{x}\right), \ \underline{\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)}$$

Relation of $\csc^{-1}x$ and $\sin^{-1}x$: let $y = \csc^{-1}x$, then $\csc y = x$

$$\frac{1}{\sin y} = x, \quad \sin y = \frac{1}{x}, \quad y = \sin^{-1}\left(\frac{1}{x}\right), \ \underline{\csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)}$$

Relation of $\cot^{-1}x$ and $\tan^{-1}x$: let $y = \cot^{-1}x$, then $\cot y = x$

$$\frac{1}{\tan y} = x, \quad \tan y = \frac{1}{x}, \quad y = \tan^{-1}\left(\frac{1}{x}\right), \ \underline{\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)}$$

2. **Derivatives of Inverse Trigonometric Functions**:

$$\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1 - x^2}}, \qquad \frac{d}{dx}[\cos^{-1}(x)] = \frac{-1}{\sqrt{1 - x^2}}, \text{ for } -1 < x < 1$$

$$\frac{d}{dx}[\tan^{-1}(x)] = \frac{1}{1 + x^2}, \qquad \frac{d}{dx}[\cot^{-1}(x)] = \frac{-1}{1 + x^2}$$

$$\frac{d}{dx}[\sec^{-1}(x)] = \frac{1}{|x|\sqrt{x^2 - 1}}, \quad \frac{d}{dx}[\csc^{-1}(x)] = \frac{-1}{|x|\sqrt{x^2 - 1}}, \text{ for } |x| > 1$$

Derivations:

(1) $\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$

Let $y = \sin^{-1}(x)$. Then $\frac{d}{dx}[\sin^{-1}(x)] = \frac{dy}{dx}$ (or y').

$$y = \sin^{-1}(x) \quad \Leftrightarrow \quad \sin(y) = x$$

Now we use the **implicit differentiation** to compute $\frac{dy}{dx}$.

$$\frac{d}{dx}[\ \sin(y)\ ] = \frac{d}{dx}(x) \quad \Leftrightarrow \quad \cos(y)\frac{dy}{dx} = 1, \qquad \frac{dy}{dx} = \frac{1}{\cos(y)}$$

$\sin(y) = x$, y is in $\left[-\frac{\pi}{2}, \ \frac{\pi}{2}\right]$, know $\cos^2(y) + \sin^2(y) = 1$

$\cos(y) = \sqrt{1-\sin^2(y)} = \sqrt{1-x^2}$ since $\cos(y) \geq 0$ for y in $\left[-\frac{\pi}{2}, \ \frac{\pi}{2}\right]$

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}, \quad \text{for } -1 < x < 1.$$

Derivations:

(2) $\frac{d}{dx}[\tan^{-1}(x)] = \frac{1}{1+x^2}$

Let $y = \tan^{-1}(x)$. Then $\frac{d}{dx}[\tan^{-1}(x)] = \frac{dy}{dx}$ (or y').

$$y = \tan^{-1}(x) \quad \Leftrightarrow \quad \tan(y) = x$$

Now we use the **implicit differentiation** to compute $\frac{dy}{dx}$.

$$\frac{d}{dx}[\ \tan(y)\ ] = \frac{d}{dx}(x) \quad \Leftrightarrow \quad \sec^2(y)\frac{dy}{dx} = 1, \quad \frac{dy}{dx} = \frac{1}{\sec^2(y)}$$

$\tan(y) = x,\ y$ is in $(-\infty,\ \infty)$,

know $\sec^2(y) - \tan^2(y) = 1,\ \sec^2(y) = 1 + \tan^2(y) = 1 + x^2$

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{1+x^2}.$$

Derivations:

(3) $\frac{d}{dx}[\sec^{-1}(x)] = \frac{1}{|x|\sqrt{x^2 - 1}}$, for $|x| > 1$

$$\frac{d}{dx}[\sec^{-1}(x)] = \frac{d}{dx}\left[\cos^{-1}\left(\frac{1}{x}\right)\right] = \frac{-1}{\sqrt{1 - (\frac{1}{x})^2}}\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x^2\sqrt{\frac{x^2 - 1}{x^2}}} \overset{\sqrt{x^2} = |x|}{=} \frac{1}{\frac{x^2}{|x|}\sqrt{\frac{x^2 - 1}{x^2}}} = \frac{1}{|x|\sqrt{x^2 - 1}}$$

Example: *Find* $f'(x)$ *where*

a. $f(x) = 2\sin^{-1}(4x^2))$ b. $f(x) = \cos^{-1}\left(\sqrt{2x^2 - 1}\right)$

c. $f(x) = \tan^{-1}(\sin(\pi x))$ d. $f(x) = \sec^{-1}(\ln(x))$

e. $f(x) = -\frac{1}{2}e^{\tan(4x)}$

a. $f(x) = 2\sin^{-1}(4x^2))$

$$f'(x) = 2\frac{1}{\sqrt{1-(4x^2)^2}}(4(2x)) = \frac{16x}{\sqrt{1-(4x^2)^2}}$$

b. $f(x) = \cos^{-1}\left(\sqrt{2x^2-1}\right)$

$$f'(x) = \frac{-1}{\sqrt{1-\left(\sqrt{2x^2-1}\right)^2}}\left(\frac{1}{2}(2x^2-1)^{-1/2}(2(2x))\right)$$

$$= \frac{-2x}{\sqrt{2x^2-1}\sqrt{1-\left(\sqrt{2x^2-1}\right)^2}}$$

c. $f(x) = \tan^{-1}(\sin(\pi x))$

$$f'(x) = \frac{1}{1+(\sin(\pi x))^2}(\cos(\pi x)\pi) = \frac{\pi\cos(\pi x)}{1+\sin^2(\pi x)}$$

d. $f(x) = \sec^{-1}(\ln(x))$

$$f'(x) = \frac{1}{|\ln x|\sqrt{(\ln x)^2 - 1}}\left(\frac{1}{x}\right) = \frac{1}{x|\ln x|\sqrt{(\ln x)^2 - 1}}$$

e. $f(x) = -\frac{1}{2}e^{\tan(4x)}$

$$f'(x) = -\frac{1}{2}e^{\tan(4x)}\left(\frac{1}{1 + (\tan(4x))^2}\sec^2(4x)(4)\right) = -\frac{2\sec^2(4x)}{1 + \tan^2(4x)}e^{\tan(4x)}$$

Example: *Find the equation of the tangent line to the curve* $y = x\sin^{-1}(x)$ *at the point where* $x = \frac{1}{2}$.

$$y' = \sin^{-1}x + x\frac{1}{\sqrt{1-x^2}},$$

$$y'\left(\frac{1}{2}\right) = \sin^{-1}\left(\frac{1}{2}\right) + \frac{1}{2}\frac{1}{\sqrt{1-(\frac{1}{2})^2}} = \frac{\pi}{6} + \frac{1}{2}\frac{1}{\sqrt{\frac{3}{4}}} = \frac{\pi}{6} + \frac{1}{\sqrt{3}}$$

$$y\left(\frac{1}{2}\right) = \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\left(\frac{\pi}{6}\right) = \frac{\pi}{12}$$

the equation of the tangent line: $y - \frac{\pi}{12} = \left(\frac{\pi}{6} + \frac{1}{\sqrt{3}}\right)\left(x - \frac{1}{2}\right)$

Example: *Let an object move along the curve* $y = \dfrac{\tan^{-1}(t)}{t}$ *for* $t \geq 1$. *Find the velocity of the object when* $t = 1$. *Determine if the object is increasing or decreasing at that time.*

$$v(t) = \frac{(1)\tan^{-1}t - t\dfrac{1}{1+t^2}}{t^2} = \frac{(1+t^2)\tan^{-1}t - t}{t^2(1+t^2)}$$

$$v(1) = \frac{2\tan^{-1}(1) - 1}{(1)(2)} = \frac{2\left(\dfrac{\pi}{4}\right) - 1}{2} = \frac{\pi - 2}{4} = 0.285 > 0$$

The object is increasing at $t = 1$ since $v(1) > 0$.