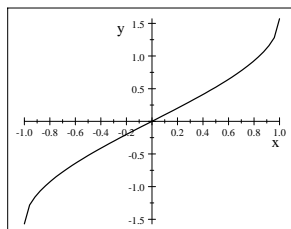


2.8 Part II - Derivatives of Inverse of Trigonometric

1. Review Inverse Trigonometric Functions:

Inverse Sine:

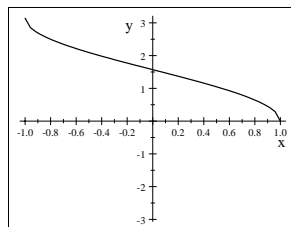
- (1) $y = \sin^{-1}(x) = \arcsin(x)$
Domain $D_{\sin^{-1}x} = [-1, 1]$
Range $R_{\sin^{-1}x} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



$$y = \sin^{-1}x, \quad x \text{ in } [-1, 1]$$

Inverse Cosine

- (2) $y = \cos^{-1}(x) = \arccos(x)$
Domain: $D_{\cos^{-1}x} = [-1, 1]$
Range: $R_{\cos^{-1}x} = [0, \pi]$

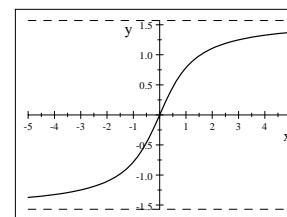


$$y = \cos^{-1}x, \quad x \text{ in } [-1, 1]$$

1

Inverse tangent

- (3) $y = \tan^{-1}x$
Domain: $D_{\tan^{-1}x} = (-\infty, \infty)$
Range: $R_{\tan^{-1}x} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$$y = \tan^{-1}x, \quad x \text{ in } (-\infty, \infty)$$

(4) Inverse Secant, Inverse Cosecant and Inverse Cotangent:

Relation of $\sec^{-1}x$ and $\cos^{-1}x$: let $y = \sec^{-1}x$, then $\sec y = x$
 $\frac{1}{\cos y} = x, \quad \cos y = \frac{1}{x}, \quad y = \cos^{-1}\left(\frac{1}{x}\right), \quad \underline{\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)}$

Relation of $\csc^{-1}x$ and $\sin^{-1}x$: let $y = \csc^{-1}x$, then $\csc y = x$

$$\frac{1}{\sin y} = x, \quad \sin y = \frac{1}{x}, \quad y = \sin^{-1}\left(\frac{1}{x}\right), \quad \underline{\csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)}$$

Relation of $\cot^{-1}x$ and $\tan^{-1}x$: let $y = \cot^{-1}x$, then $\cot y = x$

$$\frac{1}{\tan y} = x, \quad \tan y = \frac{1}{x}, \quad y = \tan^{-1}\left(\frac{1}{x}\right), \quad \underline{\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)}$$

2

2. Derivatives of Inverse Trigonometric Functions:

$$\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}[\cos^{-1}(x)] = \frac{-1}{\sqrt{1-x^2}}, \quad \text{for } -1 < x < 1$$

$$\frac{d}{dx}[\tan^{-1}(x)] = \frac{1}{1+x^2}, \quad \frac{d}{dx}[\cot^{-1}(x)] = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}[\sec^{-1}(x)] = \frac{1}{|x|\sqrt{x^2-1}}, \quad \frac{d}{dx}[\csc^{-1}(x)] = \frac{-1}{|x|\sqrt{x^2-1}}, \quad \text{for } |x| > 1$$

3

Derivations:

$$(1) \frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}, \quad \text{for } -1 < x < 1$$

Let $y = \sin^{-1}(x)$. Then $\frac{d}{dx}[\sin^{-1}(x)] = \frac{dy}{dx}$ (or y').

$$y = \sin^{-1}(x) \Leftrightarrow \sin(y) = x$$

Now we use the **implicit differentiation** to compute $\frac{dy}{dx}$.

$$\frac{d}{dx}[\sin(y)] = \frac{d}{dx}(x) \Leftrightarrow \cos(y) \frac{dy}{dx} = 1, \quad \frac{dy}{dx} = \frac{1}{\cos(y)}$$

$$\sin(y) = x, \quad y \text{ is in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad \text{know } \cos^2(y) + \sin^2(y) = 1$$

$$\cos(y) = \sqrt{1 - \sin^2(y)} = \sqrt{1 - x^2} \quad \text{since } \cos(y) \geq 0 \text{ for } y \text{ in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}, \quad \text{for } -1 < x < 1.$$

4

Derivations:

$$(2) \frac{d}{dx} [\tan^{-1}(x)] = \frac{1}{1+x^2}$$

Let $y = \tan^{-1}(x)$. Then $\frac{d}{dx} [\tan^{-1}(x)] = \frac{dy}{dx}$ (or y').

$$y = \tan^{-1}(x) \Leftrightarrow \tan(y) = x$$

Now we use the **implicit differentiation** to compute $\frac{dy}{dx}$.

$$\frac{d}{dx} [\tan(y)] = \frac{d}{dx}(x) \Leftrightarrow \sec^2(y) \frac{dy}{dx} = 1, \quad \frac{dy}{dx} = \frac{1}{\sec^2(y)}$$

$$\tan(y) = x, \text{ } y \text{ is in } (-\infty, \infty),$$

$$\text{know } \sec^2(y) - \tan^2(y) = 1, \quad \sec^2(y) = 1 + \tan^2(y) = 1 + x^2$$

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{1+x^2}.$$

5

Derivations:

$$(3) \frac{d}{dx} [\sec^{-1}(x)] = \frac{1}{|x| \sqrt{x^2 - 1}}, \text{ for } |x| > 1$$

$$\begin{aligned} \frac{d}{dx} [\sec^{-1}(x)] &= \frac{d}{dx} \left[\cos^{-1} \left(\frac{1}{x} \right) \right] = \frac{-1}{\sqrt{1 - \left(\frac{1}{x} \right)^2}} \left(-\frac{1}{x^2} \right) \\ &= \frac{1}{x^2 \sqrt{\frac{x^2 - 1}{x^2}}} \stackrel{\sqrt{x^2} = |x|}{=} \frac{1}{\frac{x^2}{|x|} \sqrt{\frac{x^2 - 1}{x^2}}} = \frac{1}{|x| \sqrt{x^2 - 1}} \end{aligned}$$

6

Example: Find $f'(x)$ where

a. $f(x) = 2 \sin^{-1}(4x^2)$ b. $f(x) = \cos^{-1}(\sqrt{2x^2 - 1})$

c. $f(x) = \tan^{-1}(\sin(\pi x))$ d. $f(x) = \sec^{-1}(\ln(x))$

e. $f(x) = -\frac{1}{2} e^{\tan(4x)}$

7

a. $f(x) = 2 \sin^{-1}(4x^2)$

$$f'(x) = 2 \frac{1}{\sqrt{1 - (4x^2)^2}} (4(2x)) = \frac{16x}{\sqrt{1 - (4x^2)^2}}$$

b. $f(x) = \cos^{-1}(\sqrt{2x^2 - 1})$

$$\begin{aligned} f'(x) &= \frac{-1}{\sqrt{1 - (\sqrt{2x^2 - 1})^2}} \left(\frac{1}{2} (2x^2 - 1)^{-1/2} (2(2x)) \right) \\ &= \frac{-2x}{\sqrt{2x^2 - 1} \sqrt{1 - (\sqrt{2x^2 - 1})^2}} \end{aligned}$$

c. $f(x) = \tan^{-1}(\sin(\pi x))$

$$f'(x) = \frac{1}{1 + (\sin(\pi x))^2} (\cos(\pi x) \pi) = \frac{\pi \cos(\pi x)}{1 + \sin^2(\pi x)}$$

8

d. $f(x) = \sec^{-1}(\ln(x))$

$$f'(x) = \frac{1}{|\ln x| \sqrt{(\ln x)^2 - 1}} \left(\frac{1}{x} \right) = \frac{1}{x |\ln x| \sqrt{(\ln x)^2 - 1}}$$

e. $f(x) = -\frac{1}{2} e^{\tan(4x)}$

$$f'(x) = -\frac{1}{2} e^{\tan(4x)} \left(\frac{1}{1 + (\tan(4x))^2} \sec^2(4x)(4) \right) = -\frac{2 \sec^2(4x)}{1 + \tan^2(4x)} e^{\tan(4x)}$$

9

Example: Find the equation of the tangent line to the curve

$y = x \sin^{-1}(x)$ at the point where $x = \frac{1}{2}$.

$$y' = \sin^{-1}x + x \frac{1}{\sqrt{1-x^2}},$$

$$y'\left(\frac{1}{2}\right) = \sin^{-1}\left(\frac{1}{2}\right) + \frac{1}{2} \frac{1}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = \frac{\pi}{6} + \frac{1}{2} \frac{1}{\sqrt{\frac{3}{4}}} = \frac{\pi}{6} + \frac{1}{\sqrt{3}}$$

$$y\left(\frac{1}{2}\right) = \frac{1}{2} \sin^{-1}\left(\frac{1}{2}\right) = \frac{1}{2} \left(\frac{\pi}{6} \right) = \frac{\pi}{12}$$

the equation of the tangent line: $y - \frac{\pi}{12} = \left(\frac{\pi}{6} + \frac{1}{\sqrt{3}} \right) \left(x - \frac{1}{2} \right)$

10

Example: Let an object move along the curve $y = \frac{\tan^{-1}(t)}{t}$ for $t \geq 1$.

Find the velocity of the object when $t = 1$. Determine if the object is increasing or decreasing at that time.

$$v(t) = \frac{(1) \tan^{-1}t - t \frac{1}{1+t^2}}{t^2} = \frac{(1+t^2) \tan^{-1}t - t}{t^2(1+t^2)}$$

$$v(1) = \frac{2 \tan^{-1}(1) - 1}{(1)(2)} = \frac{2\left(\frac{\pi}{4}\right) - 1}{2} = \frac{\pi - 2}{4} = 0.285 > 0$$

The object is increasing at $t = 1$ since $v(1) > 0$.

11