2.9 - Rolle's Theorem and The Mean Value Theorem

1. Rolle's Theorem

Theorem: Suppose that \( f(x) \) is \textit{continuous} on the interval \([a, b]\) and \textit{differentiable} on the interval \((a, b)\) and \( f(a) = f(b) \). Then there is a number \( c \) in \((a, b)\) such that \( f'(c) = 0 \).

Graphically, Rolle's Theorem says if \( f(a) = f(b) \) then the curve \( y = f(x) \) has a \textit{horizontal tangent line} at \((c, f(c))\) for some \( c \) in \((a, b)\).

For example:

\[ f(0) = f(2) = 0, \quad f'(1) = 0. \]

\( a = 0, \quad b = 2, \quad c = 1. \)

Algebraically, the value(s) of \( c \) can be found in two steps:

(i) Compute \( f'(x) \).

(ii) Set \( f'(x) = 0 \) and solve for \( x \), and \( c = x \) for \( x \) in the interval \((a, b)\).

Example: Find a value of \( c \) satisfying the conclusion of Rolle's Theorem for \( f(x) = x^3 - 3x^2 + 2x + 2 \) on the interval \([0, 1]\).

\( f \) is a polynomial so it is continuous on \([0, 1]\) and is differentiable on \((0, 1)\). Since \( f(0) = 2 \) and \( f(1) = 2 \), Rolle's Theorem is applied here.

Find \( c \):

(i) Compute \( f'(x) \):

\[ f'(x) = 3x^2 - 6x + 2. \]

(ii) Set \( f'(x) = 0 \) and then solve for \( x \):

\[ 3x^2 - 6x + 2 = 0 \]

\[ x = \frac{6 \pm \sqrt{36 - 4(3)(2)}}{2(3)} = \frac{2\left(3 \pm \sqrt{9 - 6}\right)}{6} = \frac{3 \pm \sqrt{3}}{3} = 1 \pm \frac{\sqrt{3}}{3} \]

\[ x = 1 + \frac{\sqrt{3}}{3} > 1 \text{ is outside } [0, 1] \]

\[ x = 1 - \frac{\sqrt{3}}{3} \text{ is in } [0, 1] \]

therefore \( c = 1 - \frac{\sqrt{3}}{3} = 0.42265 \)

Check the graph of \( f(x) \) for \( x \) in \([0, 1]\) :

2. Mean Value Theorem

Theorem: Suppose that \( f \) is continuous on the interval \([a, b]\) and \textit{differentiable} on the interval \((a, b)\). Then there exists a number \( c \) in \((a, b)\) such that

\[ f'(c) = \frac{f(b) - f(a)}{b - a}. \]

Observe that

(1) Rolle's Theorem is a \textit{special case} of the Mean Value Theorem.

(2) The \textit{right side} of the equation is the \textit{slope of the secant line} to the curve \( y = f(x) \) passing through points \((a, f(a))\) and \((b, f(b))\).

The \textit{left side} of the equation is the \textit{slope of the tangent line} to the curve \( y = f(x) \) at the point \((c, f(c))\).
Algebraically, we can find $c$ in three steps:

(i) Compute $f'(x)$.

(ii) Compute $d = \frac{f(b) - f(a)}{b - a}$.

(iii) Set $f'(x) = d$, solve for $x$, and $c = x$ if it is in $(a, b)$.

**Example:** Find all possible values of $c$ satisfying the conclusion of the Mean Value Theorem for $f(x) = x^3 - 3x^2 + 2x + 2$ on the interval $[0, \frac{1}{2}]$.

$f$ is a polynomial so it is continuous on $[0, \frac{1}{2}]$ and is differentiable on $(0, \frac{1}{2})$. Find $c$:

(i) Compute $f'(x) = 3x^2 - 6x + 2$.

(ii) Compute $d = \frac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0} = -\frac{1}{4}$

(iii) Set $3x^2 - 6x + 2 = -\frac{1}{4}$, solve for $x$: $3x^2 - 6x + \frac{9}{4} = 0$

$x = \frac{3}{2}, \quad x = \frac{1}{2}$

$x = \frac{3}{2}$ is outside $(0, \frac{3}{2})$ and

$x = \frac{1}{2}$ is in $(0, \frac{3}{2})$. So, $c = \frac{1}{2}$.

**Example:** Show that for any real numbers $u$ and $v$, $|\sin(u) - \sin(v)| \leq |u - v|$. In particular, $|\sin(u)| \leq |u|$ for $u \neq 0$.

Let $f(x) = \sin(x)$. By Mean-Value Theorem, there exists $c$ between $u$ and $v$ such that

$$|\sin(u) - \sin(v)| = |f'(c)(u - v)| = |\cos(c)||u - v|$$

Let $v = 0$. Then

$$|\sin(u)| = |\sin(u) - \sin(0)| \leq |u - 0| = |u|.$$

**Example:** Explain why it is not valid to use the Mean-Value Theorem for $f(x) = \frac{1}{x^3}, [-1, 1]$.

$f(x) = \frac{1}{x^3}$ is not continuous at $x = 0$. Hence, the Mean-Value Theorem cannot be applied for this problem.