3.1 - Application of Differentiation:

1. **Linear Approximation:**

Applications:

(1) What are $\sqrt{1.001}$, $\cos(0.001)$, $\ln(1.001)$ (without a calculator)?

(2) If we know the temperature is 91° yesterday and 87° today, can we predict the temperature tomorrow?

Idea:

By the Mean-Value Theorem, there exists c in (x_0, x_1) such that

$$f'(c) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Then $f(x_1) - f(x_0) = f'(c)(x_1 - x_0) \approx f'(x_0)(x_1 - x_0)$ and

$$f(x_1) \approx f(x_0) + f'(x_0)(x_1 - x_0).$$

Example: Approximate (1) $\sqrt{1.001}$, (2) $\cos(0.001)$, (3) $\ln(1.001)$, (4) $e^{0.001}$.

(1) Let $f(x) = \sqrt{x}$ and $x_0 = 1$.

$$\sqrt{1.001} \approx 1 + \frac{1}{2\sqrt{1}}(0.001) = 1.0005.$$

(2) Let $f(x) = \cos(x)$ and $x_0 = 0$.

$$\cos(0.001) = \cos(0) - \sin(0)(0.001) = \cos(0) = 1$$

(3) Let $f(x) = \ln(x)$ and $x_0 = 1$.

$$\ln(1.001) = \ln(1) + \frac{1}{1}(0.001) = 0.001.$$

(4) Let $f(x) = e^x$ and $x_0 = 0$.

$$e^{0.001} = e^0 + e^0(0.001 - 0) = 1 + 0.001 = 1.001$$

2. **Newton's Method**:

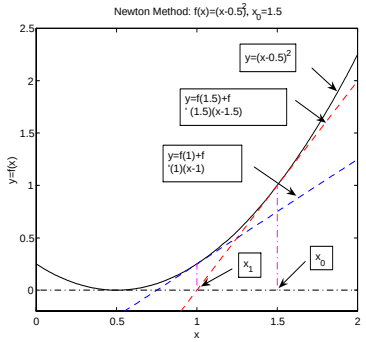
Let $y = f(x_0) + f'(x_0)(x - x_0)$. Set $y = 0$ and solve x from

$$f(x_0) + f'(x_0)(x - x_0) = 0,\ x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Continue this process.

Graphically:



Numerically:

$$x_0 = 1.5,\ f(x) = (x - 0.5)^2,\ f'(x) = 2(x - 0.5)$$

$$x_1 = 1.5 - \frac{(1.5 - 0.5)^2}{2(1.5 - 0.5)} = 1.0$$

$$x_2 = 1.0 - \frac{(1.0 - 0.5)^2}{2(1.0 - 0.5)} = 0.75\ldots$$

By calculator:

(i) type in 1.5

(ii) press 2nd+((-)/ANS) and ENTER

(iii) type in ANS-(ANS-0.5)^2/(2(ANS-0.5)) and ENTER

(iv) continue pressing ENTER untill iterations repeat within required decimal digits.

Example: Solve $x^3 + x + 1 = 0$ for x in $[-1, 0]$ by Newton's Method with $x_0 = -1$. Compute x_1 and x_2.

$$f(x) = x^3 + x + 1, \quad f'(x) = 3x^2 + 1$$

$$x_{new} = x_{old} - \frac{x_{new}^3 + x_{new} + 1}{3x_{new}^2 + 1}$$

$$x_1 = -1 - \frac{(-1)^3 + (-1) + 1}{3(-1)^2 + 1} = -\frac{3}{4}$$

$$x_2 = -\frac{3}{4} - \frac{(-\frac{3}{4})^3 + (-\frac{3}{4}) + 1}{3(-\frac{3}{4})^2 + 1} = -\frac{59}{86} = -0.686\,046\,51$$

By calculator:

$$x_3 = -0.682339582597, \quad x_4 = -0.682327803947$$