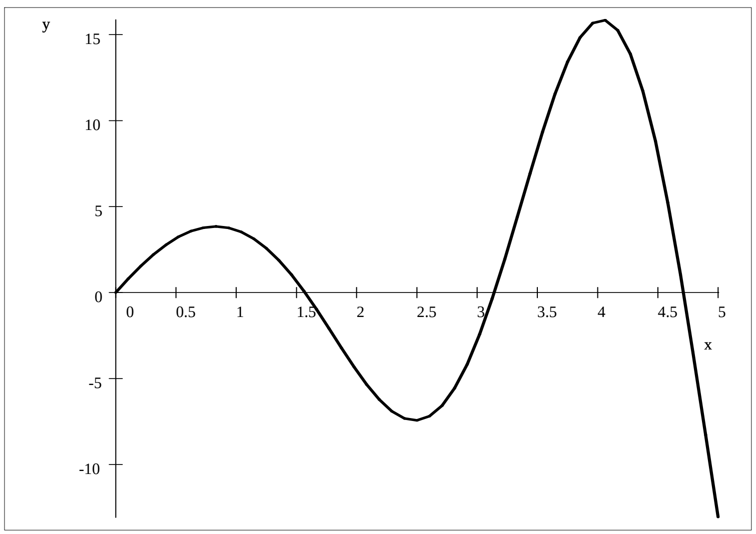
3.4 - Increasing and Decreasing Functions

1. Increasing and Decreasing Functions

Definition: *A function f is (strictly) increasing on an interval I if for every $x_1,\ x_2$ in I with $x_1 < x_2$,* $f(x_1) < f(x_2)$. *A function f is (strictly) decreasing on an interval I if for every $x_1,\ x_2$ in I with $x_1 < x_2$,* $f(x_2) < f(x_1)$.

Example: The graph of f is given below.



$f(x)$

f is increasing on $(0, 0.8), (2.5, 4)$.
f is decreasing on $(0.8, 2.5), (4, \infty)$.

Theorem: Suppose that f is differentiable on an interval I.

(i) If $f'(x) > 0$ for all x in I, then f is increasing on I.

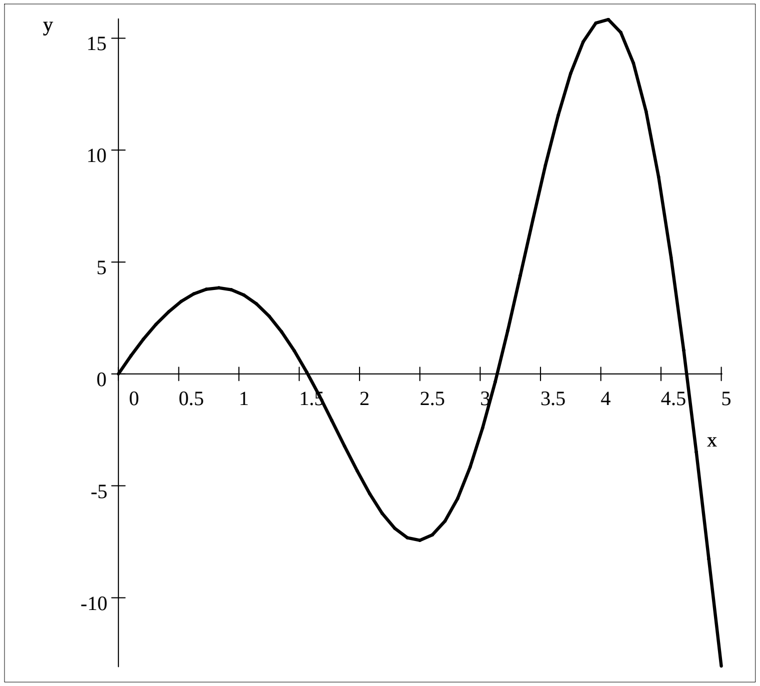
(ii) If $f'(x) < 0$ for all x in I, then f is decreasing on I.

Proof: Let x_1 and x_2 be in I and $x_1 < x_2$. Then by **the Mean Value Theorem**, we know there exists a value c in (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

If $f'(c) > 0$, then $\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$ that implies $f(x_2) - f(x_1) > 0$ because $x_2 > x_1$. Hence, $f(x_2) > f(x_1)$, that is, f is increasing. In a similar way, we can show (ii).

Example: The graph of f' is given below. Determine graphically the interval on which f is increasing.



$f'(x)$

f is increasing on $(0, 1.6), (3.1, 4.7)$
because $f'(x) > 0$

Example: *Find the intervals where* $f(x) = 2x^3 + 9x^2 - 24x - 10$ *is increasing and decreasing. Verify your answers by graphing both* f *and* f'.

Example: *Find the intervals where* $f(x) = 2x^3 + 9x^2 - 24x - 10$ *is increasing and decreasing. Verify answers with graphs of* f *and* f'.

Step I: Compute f' : $f'(x) = 6x^2 + 18x - 24 = 6(x+4)(x-1)$

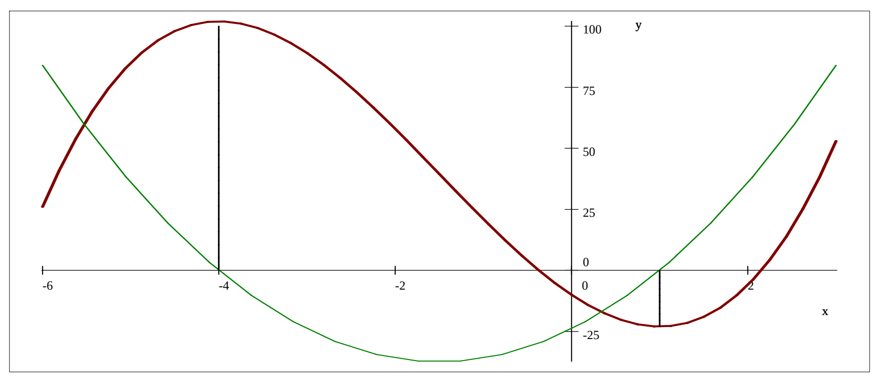
Step II: Find values of x at which $f'(x) = 0$: $x = -4$ and $x = 1$.

Step III: Check sign changes of f' over intervals: $(-\infty, -4), (-4, 1), (1, \infty)$

$$\begin{cases} f'(-5) = 36 \\ f'(0) = -24 \\ f'(2) = 36 \end{cases},$$

interval	$(-\infty, -4)$	$(-4, 1)$	$(1, \infty)$
sign of $f'(x)$	$+$	$-$	$+$

So, f is increasing on $(-\infty, -4)$, $(1, \infty)$ and is decreasing on $(-4, 1)$.



red f , green f'

2. **First Derivative Test For Local Extrema (maxima or minima):**

Theorem: Suppose that f is continuous on $[a, b]$ and c in (a, b) is a **critical number**.

(i) If f' changes sign from positive to negative at $x = c,$ then $f(c)$ is a **local maximum** of f. The number c is called **a local maximum point**.

(ii) If f' changes sign from negative to positive at $x = c,$ then $f(c)$ is a **local minimum** of f The number c is called **a local minimum point**.

(iii) If f' does not change sign at $x = c,$ then $f(c)$ is not a local extremum.

Example: *Let* $f(x) = x^2e^{-3x}$. *Find all critical numbers and use the 1st Derivative Test to classify each as the location of a local maximum, local minimum of neither.*

Example: *Let $f(x) = x^2e^{-3x}$. Find all critical numbers and use the 1st Derivative Test to classify each as the location of a local maximum, local minimum of neither.*

Step I: Find the domain of $f(x)$: $D_f = (-\infty, \infty)$.

Step II: Compute f' and find all critical numbers:

$$f'(x) = 2xe^{-3x} - 3x^2e^{-3x} = xe^{-3x}(2-3x).$$

(1) Critical number of type (i): $f'(x) = 0 \Leftrightarrow x = 0$ or $x = \frac{2}{3}$.

(2) Critical number of type (ii): None.

Step III: Check sign change of f' over intervals: $(-\infty, 0), \left(0, \frac{2}{3}\right), \left(\frac{2}{3}, \infty\right)$

$$\begin{cases} f'(-1) = (-1)e^3(5) < 0 \\ f'(\frac{1}{3}) = (\frac{1}{3})e^{-1}(1) > 0 \\ f'(1) = (1)e^{-3}(-1) < 0 \end{cases},$$

interval	$(-\infty, 0)$	$\left(0, \frac{2}{3}\right)$	$\left(\frac{2}{3}, \infty\right)$
sign of f'	$-$	$+$	$-$

$x = \frac{2}{3}$ is a local maximum point and $x = 0$ is a local minimum point.

Example: *Let* $f(x) = \frac{x^2}{x^2 - 4}$. *Find all asymptotes and extrema, and sketch the graph of* f.

Step I: horizontal and vertical asymptotes:

Horizontal asymptote: $\lim_{x \to \pm\infty} \frac{x^2}{x^2 - 4} = 1,\ y = 1$

Vertical asymptote: $x^2 - 4 = 0,\ x = 2,\ x = -2$

Step II: Compute $f'(x)$ and find all critical numbers:

$$f'(x) = \frac{2x(x^2 - 4) - x^2(2x)}{(x^2 - 4)^2} = \frac{2x(x^2 - 4 - x^2)}{(x^2 - 4)^2} = \frac{-8x}{(x^2 - 4)^2}.$$

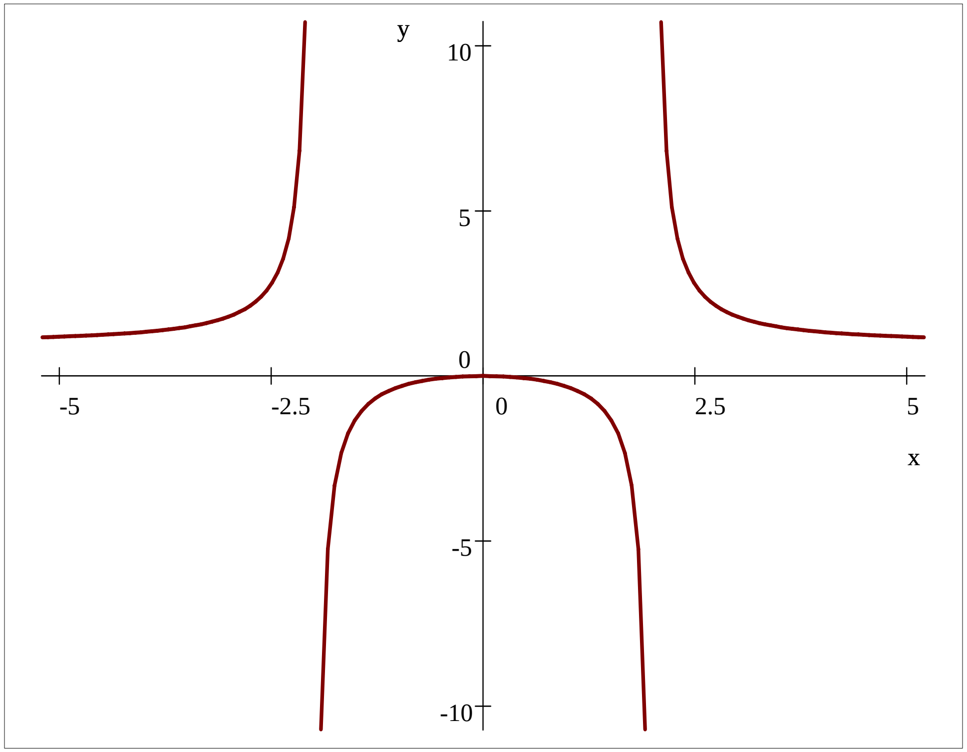
(1) Critical number of type (i): $f'(x) = 0,\ -8x = 0,\ x = 0$.

(2) Critical number of type (ii): $f'(x)$ is not defined at $x = \pm 2$ but they are not in D_f, so none.

Step III: Check sign change of f' over $(-\infty, 0),\ (0, \infty)$:

$f'(x) > 0$ for x in $(-\infty,\ 0)$ and $f'(x) < 0$ for x in $(0,\ \infty)$.

$x = 0$ is a local maximum point.



$$- \; y = f(x), \quad \text{-.-} \; y = f\,'(x)$$

Example: Let $f(x) = x^{5/3} - 3x^{2/3}$. *Find all critical numbers and use the 1st Derivative Test to classify each as the location of a local maximum, local minimum of neither.*

Step I: Find the domain of $f(x)$: $D_f = (-\infty, \infty)$.

Step II: Compute $f\,'$ and find all critical numbers:

$$f\,'(x) = \frac{5}{3}x^{2/3} - 2x^{-1/3} = x^{-1/3}\left(\frac{5}{3}x - 2\right)$$

(1) Critical number of type (i): $f\,'(x) = 0 \Leftrightarrow x = \dfrac{6}{5}$.

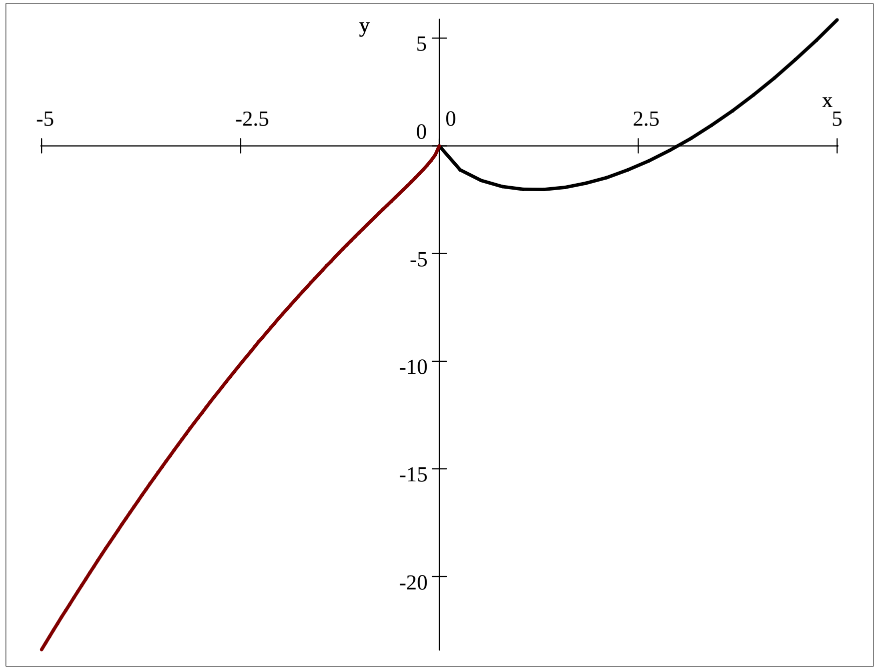
(2) Critical number of type (ii): $f\,'(x)$ is not defined when $x = 0$.

Step III: Check sign change of $f\,'$ over intervals: $(-\infty, 0), \left(0, \frac{6}{5}\right), \left(\frac{6}{5}, \infty\right)$

$$\begin{cases} f\,'(-1) = -(-\frac{11}{3}) > 0 \\ f\,'(1) = -\frac{1}{6} < 0 \\ f\,'(2) = \frac{1}{\sqrt[3]{2}}(\frac{4}{3}) > 0 \end{cases},$$

interval	$(-\infty, 0)$	$\left(0, \frac{6}{5}\right)$	$\left(\frac{6}{5}, \infty\right)$
sign of $f\,'$	$+$	-	$+$

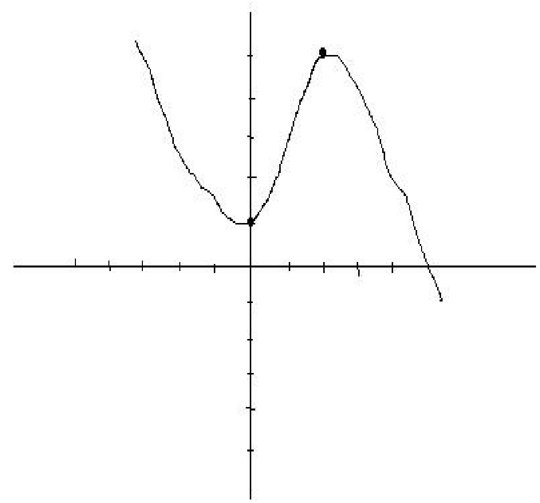
So, $x = 0$ is a local maximum point and $x = \frac{6}{5}$ is a local minimum point.



$y = f(x)$

Example: Sketch a graph of a function with the given properties:

(i) $f(0) = 1, \quad f(2) = 5$
(ii) $f'(0) = 0, \; f'(2) = 0$
$f'(x) < 0$, for $x < 0$ and $x > 2$;
$f'(x) > 0$ for $0 < x < 2$



Example: Sketch a graph of a function with the given properties:

(i) $f(-1.2) = 0, \; f(3) = 0, \; f(0)$ does
not exist
(ii) $f'(3) = 0, \; f'(0)$ does not exist
$f'(x) < 0$, for $x < 0$ and $x > 3$;
$f'(x) > 0$ for $0 < x < 3$