3.4 - Increasing and Decreasing Functions

1. Increasing and Decreasing Functions

**Definition:** A function \( f \) is (strictly) increasing on an interval \( I \) if for every \( x_1, x_2 \) in \( I \) with \( x_1 < x_2 \), \( f(x_1) < f(x_2) \). A function \( f \) is (strictly) decreasing on an interval \( I \) if for every \( x_1, x_2 \) in \( I \) with \( x_1 < x_2 \), \( f(x_2) < f(x_1) \).

**Example:** The graph of \( f \) is given below.

<table>
<thead>
<tr>
<th>( f ) is increasing on ((0, 0.8), (2.5, 4)).</th>
<th>( f ) is decreasing on ((0.8, 2.5), (4, \infty)).</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
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</table>

![Graph of f(x)](image-url)
Theorem: Suppose that $f$ is differentiable on an interval $I$.

(i) If $f'(x) > 0$ for all $x$ in $I$, then $f$ is increasing on $I$.

(ii) If $f'(x) < 0$ for all $x$ in $I$, then $f$ is decreasing on $I$.

Proof: Let $x_1$ and $x_2$ be in $I$ and $x_1 < x_2$. Then by the Mean Value Theorem, we know there exists a value $c$ in $(x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$ 

If $f'(c) > 0$, then $\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$ that implies $f(x_2) - f(x_1) > 0$ because $x_2 > x_1$. Hence, $f(x_2) > f(x_1)$, that is, $f$ is increasing. In a similar way, we can show (ii).
Example: The graph of $f'$ is given below. Determine graphically the interval on which $f$ is increasing.

\[ f' \]

\[ \begin{array}{c}
\text{f is increasing on (0, 1.6), (3.1, 4.7)} \\
\text{because } f'(x) > 0
\end{array} \]

Example: Find the intervals where $f(x) = 2x^3 + 9x^2 - 24x - 10$ is increasing and decreasing. Verify your answers by graphing both $f$ and $f'$. 

\[ f' \]
Example: Find the intervals where \( f(x) = 2x^3 + 9x^2 - 24x - 10 \) is increasing and decreasing. Verify answers with graphs of \( f \) and \( f' \).

Step I: Compute \( f' : f'(x) = 6x^2 + 18x - 24 = 6(x + 4)(x - 1) \)

Step II: Find values of \( x \) at which \( f'(x) = 0 \): \( x = -4 \) and \( x = 1 \).

Step III: Check sign changes of \( f' \) over intervals: \((-\infty,-4), (-4,1), (1,\infty)\)

\[
\begin{align*}
  f'(-5) &= 36 \\
  f'(0) &= -24 \\
  f'(2) &= 36
\end{align*}
\]

<table>
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<tr>
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<th>((-\infty,-4))</th>
<th>((-4,1))</th>
<th>((1,\infty))</th>
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<td>sign of ( f'(x) )</td>
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<td>-</td>
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So, \( f \) is increasing on \((-\infty,-4), (1,\infty)\) and is decreasing on \((-4,1)\).
2. First Derivative Test For Local Extrema (maxima or minima):

**Theorem:** Suppose that $f$ is continuous on $[a, b]$ and $c$ in $(a, b)$ is a critical number.

(i) If $f'$ changes sign from positive to negative at $x = c$, then $f(c)$ is a local maximum of $f$. The number $c$ is called a local maximum point.

(ii) If $f'$ changes sign from negative to positive at $x = c$, then $f(c)$ is a local minimum of $f$. The number $c$ is called a local minimum point.

(iii) If $f'$ does not change sign at $x = c$, then $f(c)$ is not a local extremum.

**Example:** Let $f(x) = x^2 e^{-3x}$. Find all critical numbers and use the 1st Derivative Test to classify each as the location of a local maximum, local minimum of neither.
**Example:** Let \( f(x) = x^2 e^{-3x} \). Find all critical numbers and use the 1st Derivative Test to classify each as the location of a local maximum, local minimum of neither.

**Step I:** Find the domain of \( f(x) : D_f = (-\infty, \infty) \).

**Step II:** Compute \( f' \) and find all critical numbers:

\[
f'(x) = 2xe^{-3x} - 3x^2 e^{-3x} = xe^{-3x}(2 - 3x).
\]

(1) Critical number of type (i): \( f'(x) = 0 \iff x = 0 \) or \( x = \frac{2}{3} \).

(2) Critical number of type (ii): None.

**Step III:** Check sign change of \( f' \) over intervals: \((-\infty, 0), \left(0, \frac{2}{3}\right), \left(\frac{2}{3}, \infty\right)\)

\[
\begin{align*}
f'(-1) &= (-1)e^3(5) < 0 \\
\frac{1}{3} & = (\frac{1}{3})e^{-1}(1) > 0 \\
f'(1) &= (1)e^{-3}(-1) < 0
\end{align*}
\]

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\( x = \frac{2}{3} \) is a local maximum point and \( x = 0 \) is a local minimum point.
Example: Let \( f(x) = \frac{x^2}{x^2 - 4} \). Find all asymptotes and extrema, and sketch the graph of \( f \).

**Step I:** horizontal and vertical asymptotes:

Horizontal asymptote: \( \lim_{x \to \pm\infty} \frac{x^2}{x^2 - 4} = 1, \ y = 1 \)

Vertical asymptote: \( x^2 - 4 = 0, \ x = 2, \ x = -2 \)

**Step II:** Compute \( f'(x) \) and find all critical numbers:

\[
f'(x) = \frac{2x(x^2 - 4) - x^2(2x)}{(x^2 - 4)^2} = \frac{2x(x^2 - 4 - x^2)}{(x^2 - 4)^2} = \frac{-8x}{(x^2 - 4)^2}.
\]

(1) Critical number of type (i): \( f'(x) = 0, \ -8x = 0, \ x = 0 \).

(2) Critical number of type (ii): \( f'(x) \) is not defined at \( x = \pm 2 \) but they are not in \( D_f \), so none.

**Step III:** Check sign change of \( f' \) over \( (-\infty, 0), \ (0, \infty) \):

\( f'(x) > 0 \) for \( x \) in \( (-\infty, 0) \) and \( f'(x) < 0 \) for \( x \) in \( (0, \infty) \).

\( x = 0 \) is a local maximum point.
\[-y = f(x), \quad \therefore y = f'(x)\]
**Example:** Let \( f(x) = x^{5/3} - 3x^{2/3} \). Find all critical numbers and use the 1st Derivative Test to classify each as the location of a local maximum, local minimum of neither.

**Step I:** Find the domain of \( f(x) \): \( D_f = (-\infty, \infty) \).

**Step II:** Compute \( f' \) and find all critical numbers:

\[
f'(x) = \frac{5}{3}x^{2/3} - 2x^{-1/3} = x^{-1/3} \left( \frac{5}{3}x - 2 \right)
\]

(1) Critical number of type (i): \( f'(x) = 0 \iff x = \frac{6}{5} \).

(2) Critical number of type (ii): \( f'(x) \) is not defined when \( x = 0 \).

**Step III:** Check sign change of \( f' \) over intervals: \((-\infty, 0), (0, \frac{6}{5}), (\frac{6}{5}, \infty)\)

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\[
\begin{align*}
  f'(-1) &= -(-\frac{11}{3}) > 0 \\
  f'(1) &= -\frac{1}{6} < 0 \\
  f'(2) &= \frac{1}{3\sqrt{2}}(\frac{4}{3}) > 0
\end{align*}
\]
So, $x = 0$ is a local maximum point and $x = \frac{6}{5}$ is a local minimum point.
Example: Sketch a graph of a function with the given properties:

| (i) | $f(0) = 1$, $f(2) = 5$ |
| (ii) | $f'(0) = 0$, $f'(2) = 0$ |
|      | $f'(x) < 0$, for $x < 0$ and $x > 2$; |
|      | $f'(x) > 0$ for $0 < x < 2$ |

Example: Sketch a graph of a function with the given properties:

| (i) | $f(-1.2) = 0$, $f(3) = 0$, $f(0)$ does not exist |
| (ii) | $f'(3) = 0$, $f'(0)$ does not exist |
|      | $f'(x) < 0$, for $x < 0$ and $x > 3$; |
|      | $f'(x) > 0$ for $0 < x < 3$ |