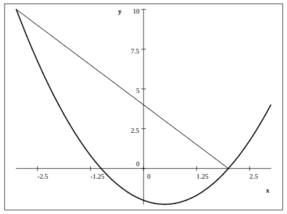
3.5 - Concavity

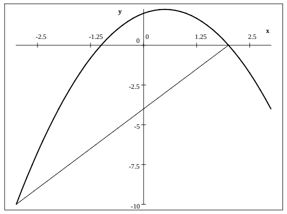
1. **Concave up and concave down**

For a function f that is differentiable on an interval I, the graph of f is

- If f is **concave up** on $[a,\ b]$, then the **secant line** passing through points $(x_1,\ f(x_1))$ and $(x_2,\ f(x_2))$ for any x_1 and x_2 in $[a,\ b]$ are **above** the curve $y = f(x)$ between $(x_1,\ f(x_1))$ and $(x_2,\ f(x_2))$.
- If f is **concave down** on $[a,\ b]$, then the **secant line** passing through points $(x_1,\ f(x_1))$ and $(x_2,\ f(x_2))$ for any x_1 and x_2 in $[a,\ b]$ are **below** the curve $y = f(x)$ between $(x_1,\ f(x_1))$ and $(x_2,\ f(x_2))$.



a concave up

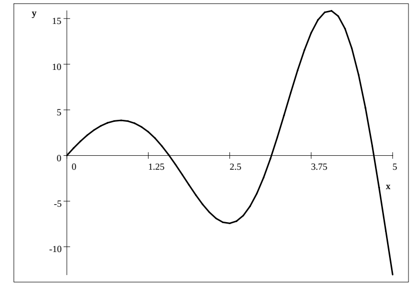


a concave down

Example: *The graph of f is given below. Determine graphically the interval on which f is*

(1)concave up;

(2) concave up and decreasing.



$f(x)$

(1) f is concave up on $(1.8,\ 3.3)$.

(2) f is concave up and decreasing on $(1.8,\ 2.5)$

How can we determine **algebraically** where f is **concave up** and where f is **concave down**?

Theorem: Suppose that f is differentiable on an interval (a,b). Then the graph of f is

(a) **concave up** on (a,b), if f' is **increasing** on (a,b) ; and

(b) **concave down** on (a,b), if f' is **decreasing** on (a,b).

Or, suppose that f'' exists on (a,b). The graph of f is

(a) **concave up** on (a,b), if $f''(x) > 0$ for all x in (a,b);

(b) **concave down** on (a,b), if $f''(x) < 0$ for all x in (a,b).

2. **Inflection points:**

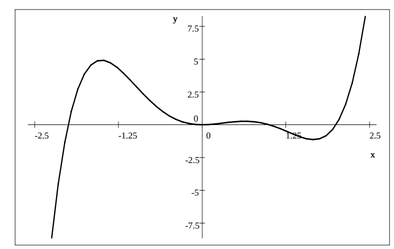
Definition: Suppose that f is continuous on the interval $(a,\ b)$. Let c be in $(a,\ b)$. Then the point $(c,\ f(c))$ is called an **inflection point** of f if the graph of f changes **concavity** at the point $(c,\ f(c))$.

Note that: **changes concavity** at $(c,\ f(c)) \Leftrightarrow f'$ changes from **increasing to decreasing** at $(c,\ f(c)) \Leftrightarrow f''$ changes from **positive to negative**or from **negative to positive** at $(c,\ f(c))$.

Example: *Let the graph of $f''(x)$ be given at right. Find*

(1) the x–coordinate of each inflection point of f;

(2) where the graph of f is concave up.



$f''(x)$

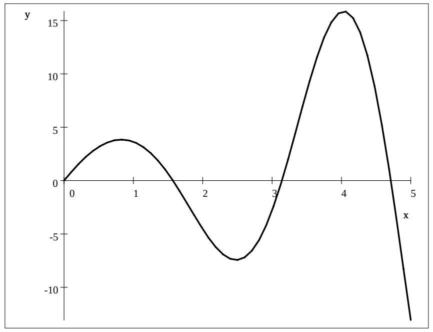
(1) $f''(x) = 0$ when $x = -2,\ x = 0,\ x = 1$ and $x = 2$. f''does not change sign at $x = 0$. So, the x–coordinates of inflection points of f are $x = -2$, $x = 1$ and $x = 2$.

(2) $f'' > 0$ for $-2 < x < 0,\ 0 < x < 1,\ 2 < x < \infty$ and $f'' < 0$ for $-\infty < x < -2, 1 < x < 2$. So, the graph of f is concave up on $(-2,0) \cup (0,1) \cup (2,\infty)$.

Example: *Let the graph of* $f'(x)$ *be given* at right. *Find*

(1) the x *–coordinate of each inflection point of f;*

(2) where the graph of f is concave up.



$f'(x)$

$f''(x) = 0$ when $x = 0.8,\ x = 2.5,\ x = 4.$

$f'' > 0$ (f' is increasing) for $0 < x < 0.8,\ 2.5 < x < 4;$

$f'' < 0$ $\left(f'\right.$ is decreasing$\left.\right)$ for $0.8 < x < 2.5,\ 4 < x < 5.$

(1) So, $x = 0.8,\ x = 2.5, x = 4$ are the x –coordinates of inflection points.

(2) The graph of f is concave up on $(0, 0.8) \cup (2.5, 4)$.

Example: *Let* $f(x) = 2x^3 + 9x^2 - 24x - 10.$ *Find*

(1) all inflection points of f;

(2) where the graph of f is concave up and is concave down.

Verify your answers by graphing both f and f''.

(1i.) Compute f'': $f'(x) = 6x^2 + 18x - 24,\ f''(x) = 12x + 18 = 12(2x+3)$

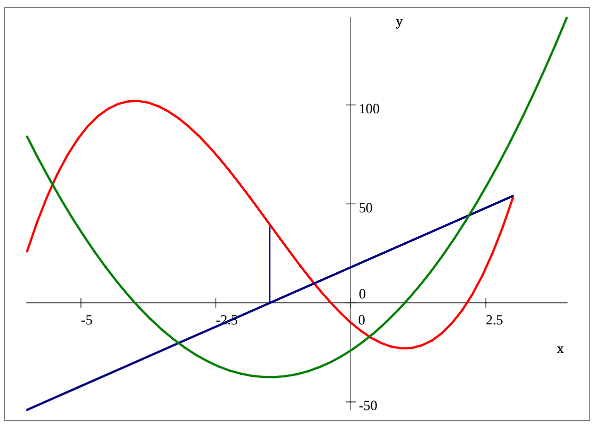
(1ii.) Solve $f''(x) = 0$: $\quad 12(2x+3) = 0 \quad \Rightarrow \quad x = -\frac{3}{2}.$

(1iii.) Check signs of f'' over intervals: $\left(-\infty, -\frac{3}{2}\right),\ \left(-\frac{3}{2}, \infty\right)$

$$\begin{cases} f''(-2) = 12(-1) < 0 \\ f''(0) = 12(3) > 0 \end{cases},$$

interval	$\left(-\infty, -\frac{3}{2}\right)$	$\left(-\frac{3}{2}, \infty\right)$
sign of $f''(x)$	$-$	$+$

Since f'' changes sign at the point where $x = -\frac{3}{2}$, $\left(-\frac{3}{2}, \frac{79}{2}\right)$ is an inflection point of *f*. The graph of *f* is concave up on $\left(-\frac{3}{2}, \infty\right)$ and is concave down on $\left(-\infty, -\frac{3}{2}\right)$. Verify the results with the graph of *f*.



red f , green f', blue f''

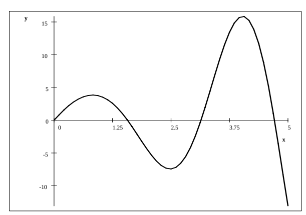
3. **Second Derivative Test:**

Theroem: Suppose that f'' is continuos on the interval $(a,\ b)$ and $f'(c) = 0,$ for some c in $(a,\ b)$.

(a) If $f''(c) < 0,$ then $f(c)$ is a local maximum and

(b) if $f''(c) > 0,$ then $f(c)$ is a local minimum.

Example: *The graph of* f'' *is given below. Suppose that we know* $f'(1) = 0,\ f'(2) = 0$ *and* $f'(4) = 0.$ *Determine if* $f(1),\ f(2)$ *and* $f(4)$ *are local maximum, local minimum or neither.*



$f''(x)$

$f''(1) > 0$ and $f''(4) > 0,$

so $f(1)$ and $f(4)$ are local maximum values.

$f''(2) < 0,\ f(2)$ is a local minimum value.

Example: *Let* $f(x) = x + 2(1-x)^{1/3}$. *Find*

(1) the intervals of increase and decrease;

(2) all local extrema;

(3) the intervals of concavity;

(4) all inflection points; and

(5) sketch the graph of f based on the information in a.-d.

The domain of f : $(-\infty,\ \infty)$

Compute f' and f'' :

$$f'(x) = 1 - \frac{2}{3}(1-x)^{-2/3} = 1 - \frac{2}{3\sqrt[3]{(1-x)^2}}$$

$$f''(x) = -\frac{4}{9}(1-x)^{-5/3} = -\frac{4}{9\sqrt[3]{(1-x)^5}}$$

Find critical numbers of f' :

type (i): $f'(x) = 0$

$$1 - \frac{2}{3\sqrt[3]{(1-x)^2}} = 0,\quad \sqrt[3]{(1-x)^2} = \frac{2}{3},\ 1 - x = \pm\sqrt{\left(\frac{2}{3}\right)^3}$$

$$x = 1 \mp \sqrt{\left(\frac{2}{3}\right)^3},\ x = 1 - \sqrt{\left(\frac{2}{3}\right)^3} = 0.4557,\ x = 1 + \sqrt{\left(\frac{2}{3}\right)^3} = 1.544$$

type (ii): $f'(x)$ is not defined : $x = 1$.

Determine the sign change of f' over

$$(-\infty, 0.4557), (0.4557, 1), (1, 1.544), (1.544, \infty)$$

$$f'(0) = 1 - \frac{2}{3\sqrt[3]{(1)^2}} = -\frac{1}{3},\ f'\left(\frac{1}{2}\right) = 1 - \frac{2}{3\sqrt[3]{(\frac{1}{2})^2}} = -0.058267$$

$$f'(1.1) = 1 - \frac{2}{3\sqrt[3]{(-0.1)^2}} = -2.094,\ f'(2) = 1 - \frac{2}{3\sqrt[3]{(-1)^2}} = \frac{1}{3}$$

interval	$(-\infty, 0.4557)$	$(0.4557, 1)$	$(1, 1.544)$	$(1.544, \infty)$
$f'(x)$	+	−	−	+

Find where $f''(x) = 0$: $-\dfrac{4}{9\sqrt[3]{(1-x)^5}} \neq 0.$

Find where $f''(x)$ is not defined: $x = 1$

Determine the sign change of f'' over intervals: $(-\infty, 1)$ and $(1, \infty)$

$$\begin{cases} f''(0) = -\dfrac{4}{9\sqrt[3]{(1)^5}} < 0 \\ f''(2) = -\dfrac{4}{9\sqrt[3]{(-1)^5}} > 0 \end{cases}$$

interval	$(-\infty, 1)$	$(1, \infty)$
$f'(x)$	−	+

State the results:

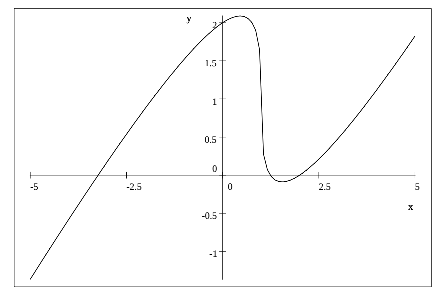
(1) f is increasing on $(-\infty, 0.4557)$, $(1.544, \infty)$ and is decreasing on $(0.4557, 1), (1, 1.544)$.

(2) By the first derivative test, $f(0.4557) = 2.0887$ is a local maximum and $f(1.544) = -0.0887$ is a local minimum.

(3) f is concave up on $(1, \infty)$ and is concave down on $(-\infty,\ 1)$.

(4) f changes concavity at $x = 1$ and $x = 1$ is in the domain of f so it is an inflection point of f.

(5) Sketch the graph of f based on the information in a.-d



Example: *Let* $f(x) = e^{-x}\cos x$. *Find*

(1) the intervals of increase and decrease;

(2) all local extrema;

(3) the intervals of concavity;

(4) all inflection points; and

(5) sketch the graph of f based on the information in a.-d.

The domain of f : $D_f = (-\infty,\ \infty)$

Compute f' and f'' :

$$f'(x) = -e^{-x}\cos x - e^{-x}\sin x = -e^{-x}(\cos x + \sin x)$$

$$f''(x) = e^{-x}(\cos x + \sin x) - e^{-x}(-\sin x + \cos x) = 2e^{-x}\sin x$$

Find critical numbers of f' :

type (i): $f'(x) = 0$

$$-e^{-x}(\cos x + \sin x) = 0,\ \cos x + \sin x = 0,\ \sin x = -\cos x,\ \tan x = -1$$

$$x = -\frac{\pi}{4} \pm n\pi,\ n = 0,\ 1,\ 2,\ \ldots$$

type (ii): $f'(x)$ is not defined: None.

Determine the sign change of f' over

$$\ldots\left(-\frac{9\pi}{4},-\frac{5\pi}{4}\right),\left(-\frac{5\pi}{4},-\frac{\pi}{4}\right),\left(-\frac{\pi}{4},\frac{3\pi}{4}\right),\left(\frac{3\pi}{4},\frac{7\pi}{4}\right),\left(\frac{7\pi}{4},\frac{11\pi}{4}\right),\ \ldots$$

$$f'(-2\pi) = e^{2\pi}\cos(-2\pi) = e^{2\pi} > 0,\ f'(-\pi) = e^{\pi}\cos(-\pi) = 0 = -e^{\pi} < 0$$

$$f'(0) = 1,\ f'(\pi) = e^{-\pi}\cos(\pi) = -e^{-\pi} < 0,\ f'(2\pi) = e^{-2\pi}\cos(2\pi) = e^{-2\pi} > 0$$

interval	$\left(-\frac{9\pi}{4},-\frac{5\pi}{4}\right)$	$\left(-\frac{5\pi}{4},-\frac{\pi}{4}\right)$	$\left(-\frac{\pi}{4},\frac{3\pi}{4}\right)$	$\left(\frac{\pi}{4},\frac{5\pi}{4}\right)$	$\left(\frac{5\pi}{4}\right.$
sign of f'	+	–	+	–	-

Find where $f''(x) = 0$: $2e^{-x}\sin x = 0,\ x = \pm n\pi,\ n = 0, 1, 2, \ldots$

Find where $f''(x)$ is not defined: None.

Determine the sign change of $f''(x) = 2e^{-x}\sin x$ over intervals:

$$\ldots(-2\pi,-\pi),(-\pi,0),(0,\pi),(\pi,2\pi),\ \ldots$$

$$f''\left(-\frac{3}{2}\pi\right) = 2e^{3\pi/2}\sin\left(-\frac{3}{2}\pi\right) > 0,\ f''\left(-\frac{1}{2}\pi\right) = 2e^{\pi/2}\sin\left(-\frac{1}{2}\pi\right) < 0$$

$$f''\left(\frac{1}{2}\pi\right) = 2e^{-\pi/2}\sin\left(\frac{1}{2}\pi\right) > 0,\ f''\left(\frac{3}{2}\pi\right) = 2e^{\pi/2}\sin\left(\frac{3}{2}\pi\right) < 0$$

interval	$(-2\pi,-\pi)$	$(-\pi,0)$	$(0,\pi)$	$(\pi,2\pi)$
sign of f''	+	–	+	–

State the results:

(1) f is increasing on ... $\left(-\frac{9\pi}{4},-\frac{5\pi}{4}\right),\ \left(-\frac{\pi}{4},\frac{3\pi}{4}\right),\ \left(\frac{5\pi}{4},\frac{9\pi}{4}\right),\ \ldots$

(2) $f(c)$ is a local maximum for $c = -\frac{5\pi}{4},\ \frac{3\pi}{4},\ \ \frac{9\pi}{4},\ \ldots$

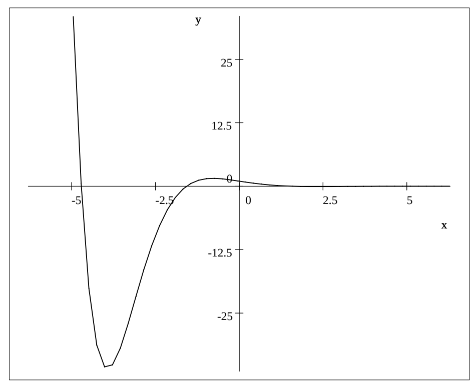
$f(c)$ is a local minimum for $c = \ldots\ -\frac{\pi}{4},\ \frac{5\pi}{4},\ \ldots$

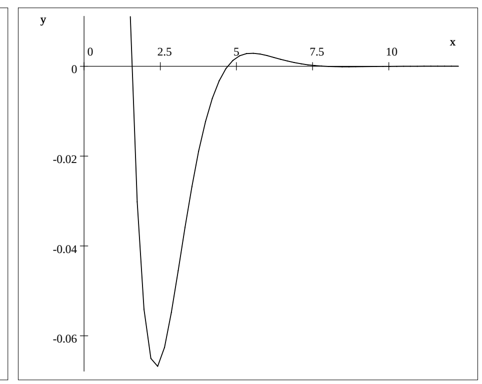
(3) f is concave up on ... $(-2\pi,-\pi),(0,\pi),\ \ldots$

and is concave down on $\ldots(-\pi,0),\ (\pi,2\pi)\ldots$

(4) Inflection points of f are: $\pm n\pi$

(5) Sketch the graph of f based on the information in (1)-(4)





Example: Sketch a graph of a function with the given properties:

(i) $f(0) = 2$

(ii) $f'(x) > 0$, for all x; $f'(0) = 1$

(iii) $f''(x) > 0$ for $x > 0$,

$f''(x) < 0$ for $x < 0$, $f''(0) = 0$

Example: Sketch a graph of a function with the given properties:

(i) $f(0) = 0$, $f(-1) = 1$, $f(1) = 1$

(ii) $f'(x) > 0$, for $x < -1$ and $0 < x < 1$,

$f'(x) < 0$ for $-1 < x < 0$ and $x > 1$;

(iii) $f''(x) < 0$ for $x < 0$ and $x > 0$