

3.7 - Minimization and Maximization Problems

1. Absolute Minimum and Absolute Maximum:

Let f be continuous on $[a, b]$. Then f **must attain** its absolute maximum and minimum values on $[a, b]$. Furthermore, if f is also differentiable on (a, b) . Then $f(c)$ is **the absolute maximum** (**absolute minimum**) if c is a **critical number** in (a, b) or $c = a$ or $c = b$.

Steps to find absolute maximum and minimum of $f(x)$ on $[a, b]$:

1. **Find** all critical numbers of $f(x)$ in (a, b) , say, c_1, \dots, c_k .
2. **Compare** values of $f(c_1), f(c_2), \dots, f(c_k), f(a)$ and $f(b)$, and
 - a. the **maximum value** of these numbers is the maximum value of f ; and
 - b. the **minimum value** of these numbers is the minimum value of f .

Example: Let $f(x) = -2x^3 + 3x^2 + 12x + 5$. Find the absolute extrema of $f(x)$ on (i) $[0, 4]$ and (ii) $[-2, 1]$.

Compute $f'(x)$: $f'(x) = -6x^2 + 6x + 12 = -6(x - 2)(x + 1)$

Critical numbers of $f(x)$: $x = 2, x = -1$.

(i) Critical number $x = 2$ is in $[0, 4]$. Check values of $f(2)$, $f(0)$ and $f(4)$:

$$f(2) = -2(2)^3 + 3(2)^2 + 12(2) + 5 = 25$$

$$f(0) = 5,$$

$$f(4) = -2(4)^3 + 3(4)^2 + 12(4) + 5 = -27$$

Hence, the absolute maximum $(2, 25)$ and absolute minimum $(4, -27)$.

(ii) Critical number $x = -1$ is $[-2, 1]$. Check values of $f(-1)$, $f(-2)$ and $f(1)$:

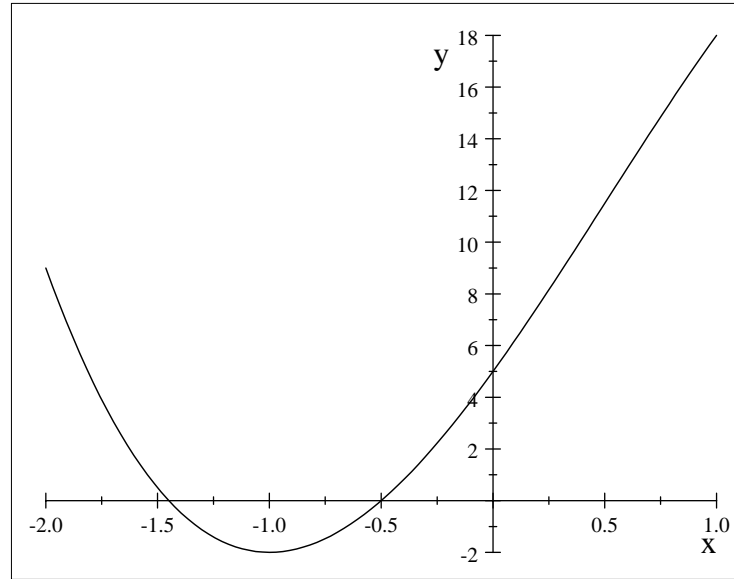
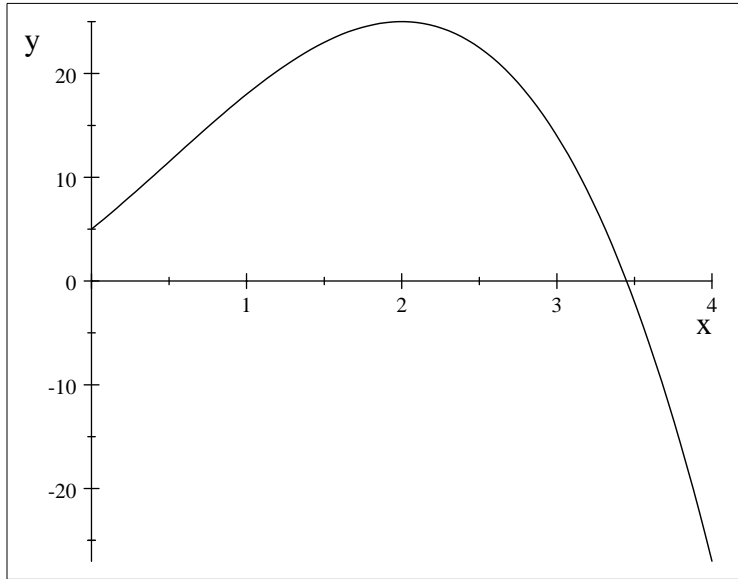
$$f(-1) = -2(-1)^3 + 3(-1)^2 + 12(-1) + 5 = -2$$

$$f(-2) = -2(-2)^3 + 3(-2)^2 + 12(-2) + 5 = 9$$

$$f(1) = -2(1)^3 + 3(1)^2 + 12(1) + 5 = 18$$

Hence, the absolute maximum $(1, 18)$ and absolute minimum $(-1, -2)$.

Verify by the graphs:



Applications in Minimization and Maximization:

Set up $f(x)$ and interval $[a, b]$, and then solve the problem.

Example: *A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?*

Let w and h be the **width** and the **height** of the field. Then the total fencing material is $2w + h = 2400$, and

$$h = 2400 - 2w = 2(1200 - w), \quad 0 \leq w \leq 1200;$$

The area $A(w) = wh = 2w(1200 - w)$, $0 \leq w \leq 1200$

We want to maximize $A(w)$ for w in $[0, 1200]$.

1. Compute $A'(w)$: $A'(w) = 2(1200 - w + w(-1)) = 4(600 - w)$
2. Find critical numbers of $A(w)$ in $(0, 1200)$:

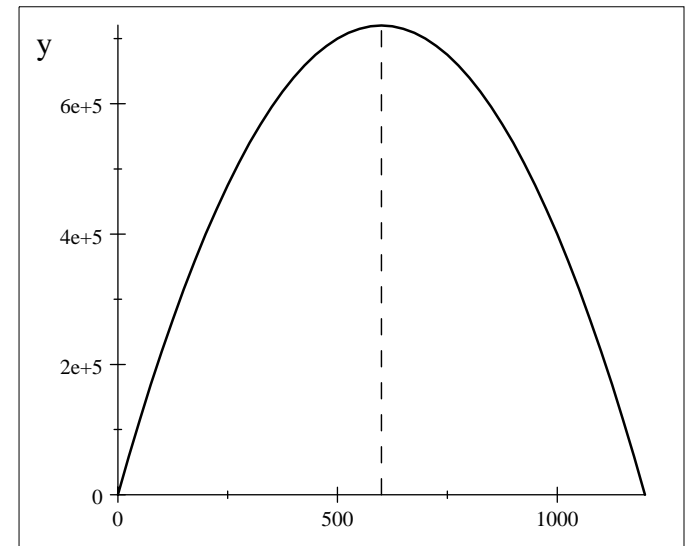
$$A'(w) = 0 \iff 4(600 - w) = 0, \quad w = 600.$$

3. Compare values of A at $w = 0$, $w = 600$ and $w = 1200$:

$$A(0) = 0, \quad A(600) = 2(600)(1200 - 600) = 720\,000, \quad A(1200) = 0$$

Hence, $A(600) = 720\,000$ is the maximum area of A .

Verify the answer with the graph of $A(w)$:



$$y = A(w), \quad 0 \leq w \leq 1200$$

Example: A sheet of $12'' \times 10''$ cardboard is made into an open box (no top) by cutting squares of equal size out of each corner and then folding up the sides. Find the dimensions of the box with the maximum volume.

Let the **size of the squares** being cut be x . Let V be the **volume** of the box. Then $0 < 2x < 10$, and $0 < 2x < 12$. So $0 < x < 5$

$$V(x) = lwh = (10 - 2x)(12 - 2x)x = 4(5 - x)(6 - x)x, \quad 0 < x < 5.$$

We want to maximize $V(x)$ for x in $[0, 5]$.

1. Compute $V'(x)$:

$$\begin{aligned} V'(x) &= 4(-6 - x)x - (5 - x)x + (5 - x)(6 - x) \\ &= 4(-6x + x^2 - 5x + x^2 + 30 - 11x + x^2) = 4(3x^2 - 22x + 30) \end{aligned}$$

2. Find critical numbers of $V(x)$ in $(0, 5)$:

$$V'(x) = 0 \iff 3x^2 - 22x + 30 = 0, \text{ Solution is:}$$

Use the quadratic formula to solve the equation:

$$x = \frac{22 \pm \sqrt{22^2 - 4(3)(30)}}{2(3)} = \frac{22 \pm \sqrt{124}}{6} = \frac{11 \pm \sqrt{31}}{3}$$

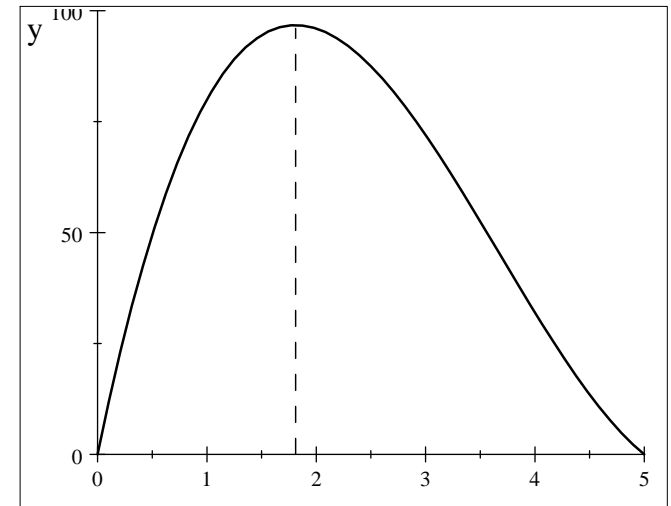
$$x = \frac{11 + \sqrt{31}}{3} = 5.523 \text{ outside } (0, 5), \quad x = \frac{11 - \sqrt{31}}{3} = 1.811$$

Compare values of V at $x = 0$, $x = 2.26$ and $x = 5$:

$$V(0) = 0, \quad V(1.811) = 96.771, \quad V(5) = 0.$$

So, $V(1.811) = 96.771 \text{ inch}^3$ is the maximum volume.

Verify the answer with the graph of V :



$$y = V(x)$$

Example: Find a point on the parabola $y = 9 - x^2$ closest to the point $(3, 8)$.

Let (x, y) be a point on the parabola $y = 9 - x^2$. Then

$$(x, y) = (x, 9 - x^2)$$

and the distance D between $(x, 9 - x^2)$ and the point $(3, 8)$ is:

$$D(x) = \sqrt{(x - 3)^2 + (9 - x^2 - 8)^2} = \sqrt{(x - 3)^2 + (1 - x^2)^2}, \quad 0 \leq x \leq 5$$

Minimizing $D(x)$ is equivalent to minimizing

$$f(x) = D^2(x) = (x - 3)^2 + (1 - x^2)^2, \quad 0 \leq x \leq 5$$

Now we want to minimize $f(x)$ for x in $[0, 5]$

1. Compute

$$f'(x) = 2(x - 3) + 2(1 - x^2)(-2x) = 2(x - 3 - 2x + 2x^3) = 2(2x^3 - x - 3)$$

2. Find all critical numbers of $f(x)$: $f'(x) = 0 \iff 2x^3 - x - 3 = 0$.

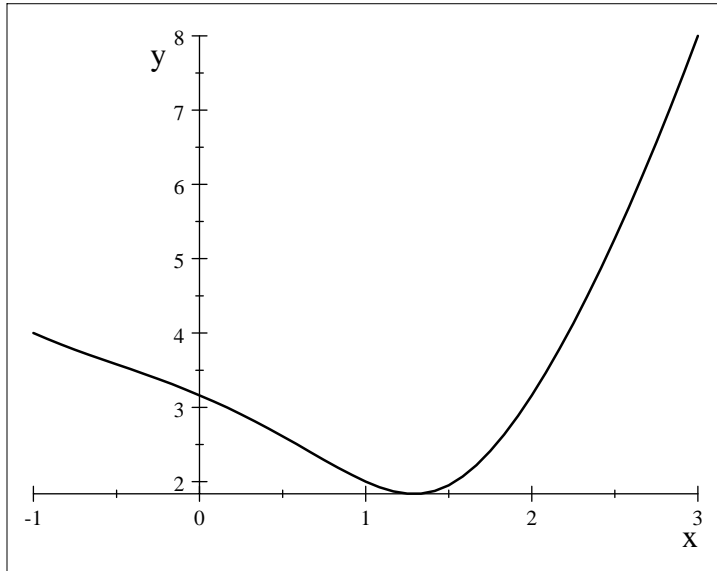
Real solution: $x = 1.290$.

3. Compare values of $f(x)$ at $x = 0$, $x = 5$ and $x = 1.29$:

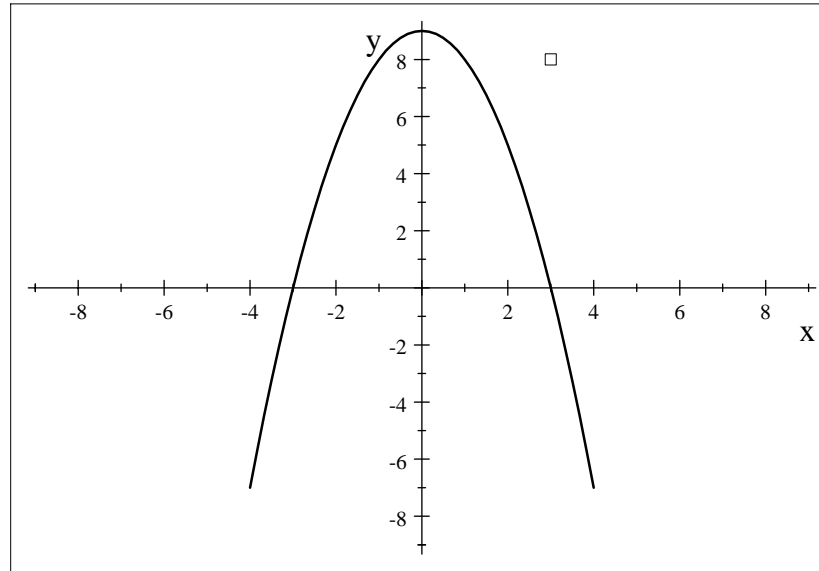
$$f(0) = 10.0, \quad f(5) = 580.0, \quad f(1.29) = 3.365.$$

Hence, $D(1.29) = \sqrt{3.36512881} = 1.834$ is the minimum distance from the parabola to the point $(3, 8)$.

Verify the answer with the graph of $D(x)$ and the parabola:



$$y = D(x)$$



$$y = 9 - x^2$$

Example: Find a point on the curve $y = \sin(x)$ closest to the point $(0, 1)$.

Let (x, y) be a point on the curve $y = \sin(x)$. Then

$$(x, y) = (x, \sin(x))$$

and the distance D between $(x, \sin(x))$ and the point $(0, 1)$ is:

$$D(x) = \sqrt{x^2 + (\sin(x) - 1)^2} = \sqrt{x^2 + \sin^2(x) - 2\sin(x) + 1}, \quad -1 \leq x \leq 1$$

Minimizing $D(x)$ is equivalent to minimizing

$$f(x) = D^2(x) = x^2 + \sin^2(x) - 2\sin(x) + 1, \quad -1 \leq x \leq 1$$

We want to minimize $f(x)$ for x in $[-1, 1]$.

1. Compute $f'(x) = 2(x + \sin(x)\cos(x) - \cos(x))$
2. Find all critical numbers of $f(x)$:

$$f'(x) = 0 \iff x + \sin(x)\cos(x) - \cos(x) = 0,$$

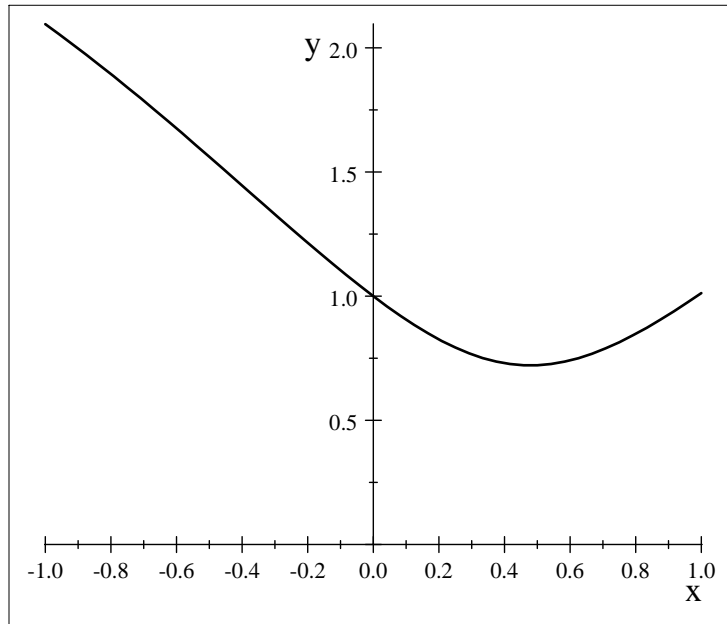
the solution is: $x = 0.479$.

3. Compare values of $f(x)$ at $x = -1$, $x = 1$ and $x = 0.479$:

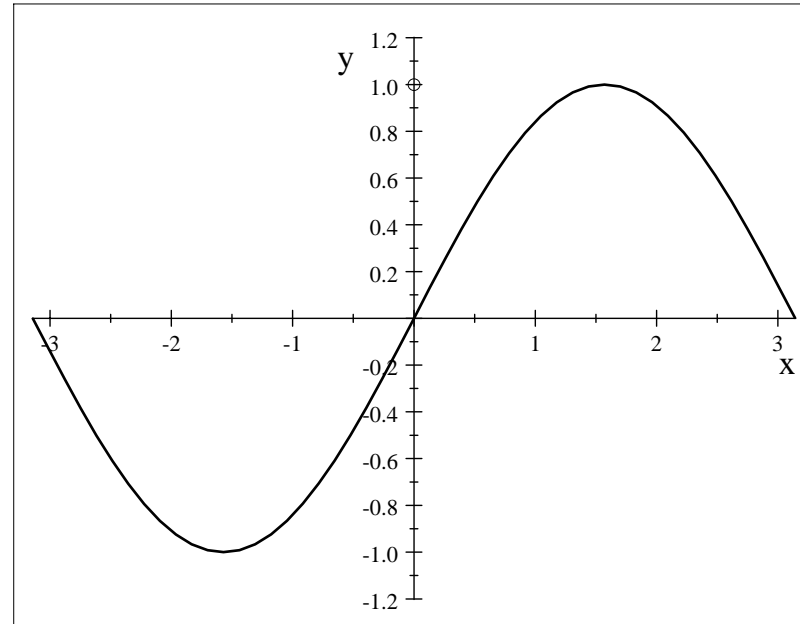
$$f(-1) = 4.391, \quad f(1) = 1.025, \quad f(0.479) = 0.520.$$

Hence, $D(0.479) = \sqrt{0.5201} = 0.7212$ is the minimum distance from the parabola to the point $(0, 1)$.

Verify the answer with the graph of $D(x)$ and the curve $y = \sin(x)$:



$$y = D(x)$$



$$y = \sin(x)$$