

## 3.8 - Related Rates

What are related rates?

**Example:** If we are pumping air into a balloon, both the **volume** and the **radius** of the balloon are **increasing** with respect to time  $t$ . Let  $V(t)$  and  $r(t)$  be the volume and the radius of the balloon at the time  $t$ . Find the relation of the relative rates  $\frac{dV}{dt}$  and  $\frac{dr}{dt}$ . If we know the rate of change of  $r$  is 0.25 inch/second when  $r = 3$  inches, what is the rate of change of the volume at the moment?

We know the relation of  $V$  and  $r$ :

$$V = \frac{4}{3}\pi r^3.$$

Implicitly,  $V = V(t)$  and  $r = r(t)$ . Then the relation of  $\frac{dV}{dt}$  and  $\frac{dr}{dt}$ :

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}, \text{ and}$$

$$\left. \frac{dV}{dt} \right|_{\text{when } r=3, \frac{dr}{dt}=0.25} = 4\pi(3)^2(0.25) = 28.27433 \text{ cubic inches/second}$$

## Example:

Assume that the infected area of an injury is circular. If the radius of the infected area is 1 mm and growing at a rate of 2 mm/hr, at what rate is the infected area increasing?

Let  $A$  be the infected area of the injury and  $r$  be the radius. Then the relation of  $A$  and  $r$  is:

$$A = \pi r^2$$

and the relation of  $\frac{dA}{dt}$  and  $\frac{dr}{dt}$  is:

$$\frac{dA}{dt} = \pi \left( 2r \frac{dr}{dt} \right) = 2\pi r \frac{dr}{dt}.$$

When  $r = 1$  and  $\frac{dr}{dt} = 2$ ,

$$\frac{dA}{dt} = 2\pi(1)(2) = 4\pi \text{ mm}^2/\text{hr}.$$

## Example:

Oil spills out of a tanker at the rate of 150 gallons per minute. The oil spreads in a circle with a thickness of  $\frac{1}{10}$  ". Determine the rate of change of the radius when the radius reaches 500 ft. ( $1 \text{ ft}^3 = 7.5$  gallons and  $1'' = \frac{1}{12}$  ft (.083 ft)).

Let  $V$  be the volume of the oil from tanker and  $r$  be the radius of the circle with a thickness of  $\frac{1}{10}$  " (oil outside tanker). Note that the volume of oil outside of tanker is the same as it is inside.

We know  $\frac{dV}{dt} = 150 \text{ gal/min} = \frac{150}{7.5} = 20 \text{ ft}^3/\text{min}$  and

the relation of  $V$  and  $r$  is:  $V = \pi r^2 \left( \frac{1}{10(12)} \right) = \frac{\pi}{120} r^2$  and

the relation of  $\frac{dV}{dt}$  and  $\frac{dr}{dt}$  is:  $\frac{dV}{dt} = \frac{\pi}{120} (2r) = \frac{\pi}{60} r \frac{dr}{dt}$ . Hence,

$\frac{dr}{dt} = \frac{60}{\pi r} \frac{dV}{dt}$ . When  $r = 500$  ft,

$$\frac{dr}{dt} = \frac{60}{\pi(500)} 20 = \frac{12}{5\pi} = 0.7694 \text{ ft/min}$$

## Example:

A 10-foot ladder leans against the side of a building. If the top of the ladder begins to slide down the wall at the rate of 2ft/sec, how fast is the bottom of the ladder sliding away from the wall when the top of the ladder is 8 ft of the ground?

Let  $x(t)$  be the distance from the bottom of the ladder to the wall horizontally and  $y(t)$  be the distance from the top of the ladder to the floor vertically.

(1) Relation of  $x(t)$  and  $y(t)$  :  $x^2 + y^2 = 10^2$

(2) Relation of  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$  :  $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ .

(3) Know that  $\frac{dy}{dt} = -2$  and  $y = 8$ , find  $\frac{dx}{dt}$ .

$$\frac{dx}{dt} = -\frac{2y}{2x} \frac{dy}{dt} = -\frac{y}{x} \frac{dy}{dt} = -\frac{8}{\sqrt{10^2-8^2}} (-2) = 2.67 \text{ ft/sec}$$