3.8 - Related Rates

What are related rates?

Example: If we are pumping air into a balloon, both the **volume** and the **radius** of the balloon are **increasing** with respect to time t. Let $V(t)$ and $r(t)$ be the volume and the radius of the balloon at the time t. Find the relation of the relative rates $\frac{dV}{dt}$ and $\frac{dr}{dt}$. If we know the rate of change of r is 0.25 inch/second when $r = 3$ inches, what is the rate of change of the volume at the moment?

We know the relation of V and r:

$$V = \frac{4}{3}\pi r^3.$$

Implicitly, $V = V(t)$ and $r = r(t)$. Then the relation of $\frac{dV}{dt}$ and $\frac{dr}{dt}$:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}, \text{ and}$$

$$\frac{dV}{dt}\Big|_{\text{when } r=3,\ \frac{dr}{dt}=0.25} = 4\pi(3)^2(0.25) = 28.27433 \text{ cubic inches/second}$$

Example:

Assume that the infected area of an injury is circular. If the radius of the infected area is 1 mm and growing at a rate of 2 mm/hr, at what rate is the infected area increasing?

Let A be the infected area of the injury and r be the radius. Then the relation of A and r is:

$$A = \pi r^2$$

and the relation of $\frac{dA}{dt}$ and $\frac{dr}{dt}$ is:

$$\frac{dA}{dt} = \pi\left(2r\frac{dr}{dt}\right) = 2\pi r\frac{dr}{dt}.$$

When $r = 1$ and $\frac{dr}{dt} = 2$,

$$\frac{dA}{dt} = 2\pi(1)(2) = 4\pi \text{ mm}^2\text{/hr}.$$

Example:

Oil spills out of a tanker at the rate of 150 gallons per minute. The oil spreads in a circle with a thickness of $\frac{1}{10}''$. Determine the rate of change of the radius when the radius reaches 500 ft. (1 ft^3 $= 7.5$ gallons and $1'' = \frac{1}{12}$ ft (.083 ft)).

Let V be the volume of the oil from tanker and r be the radius of the circle with a thickness of $\frac{1}{10}''$ (oil outside tanker). Note that the volume of oil outside of tanker is the same as it is inside.

We know $\frac{dV}{dt} = 150$ gal/min $= \frac{150}{7.5} = 20$ ft^3/min and the relation of V and r is: $V = \pi r^2\left(\frac{1}{10(12)}\right) = \frac{\pi}{120}r^2$ and the relation of $\frac{dV}{dt}$ and $\frac{dr}{dt}$ is: $\frac{dV}{dt} = \frac{\pi}{120}(2r) = \frac{\pi}{60}r\frac{dr}{dt}$. Hence, $\frac{dr}{dt} = \frac{60}{\pi r}\frac{dV}{dt}$. When $r = 500$ ft,

$$\frac{dr}{dt} = \frac{60}{\pi(500)}20 = \frac{12}{5\pi} = 0.7694 \text{ ft/min}$$

Example:

A 10-foot ladder leans against the side of a building. If the top of the ladder begins to slide down the wall at the rate of 2ft/sec, how fast is the bottom of the ladder sliding away from the wall when the top of the ladder is 8 ft of the ground?

Let $x(t)$ be the distance from the bottom of the ladder to the wall horizontally and $y(t)$ be the distance from the top of the ladder to the floor vertically.

(1) Relation of $x(t)$ and $y(t)$: $x^2 + y^2 = 10^2$

(2) Relation of $\frac{dx}{dt}$ and $\frac{dy}{dt}$: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$

(3) Know that $\frac{dy}{dt} = -2$ and $y = 8$, find $\frac{dx}{dt}$.

$$\frac{dx}{dt} = -\frac{2y}{2x}\frac{dy}{dt} = -\frac{y}{x}\frac{dy}{dt} = -\frac{8}{\sqrt{10^2-8^2}}(-2) = 2.67 \text{ ft/sec}$$