3.8 - Related Rates

What are related rates?

Example: If we are pumping air into a balloon, both the volume and the radius of the balloon are increasing with respect to time \( t \). Let \( V(t) \) and \( r(t) \) be the volume and the radius of the balloon at the time \( t \). Find the relation of the relative rates \( \frac{dV}{dt} \) and \( \frac{dr}{dt} \). If we know the rate of change of \( r \) is 0.25 inch/second when \( r = 3 \) inches, what is the rate of change of the volume at the moment?

We know the relation of \( V \) and \( r \):

\[
V = \frac{4}{3} \pi r^3
\]

Implicitly, \( V = V(t) \) and \( r = r(t) \). Then the relation of \( \frac{dV}{dt} \) and \( \frac{dr}{dt} \):

\[
\frac{dV}{dt} = 4 \pi r^2 \frac{dr}{dt}
\]

and

\[
\left. \frac{dV}{dt} \right|_{r=3, \frac{dr}{dt}=0.25} = 4 \pi (3)^2 (0.25) = 28.27433 \text{ cubic inches/second}
\]

Example:

Oil spills out of a tanker at the rate of 150 gallons per minute. The oil spreads in a circle with a thickness of \( \frac{1}{10} \)". Determine the rate of change of the radius when the radius reaches 500 ft. (1 ft\(^3\) = 7.5 gallons and 1" = \( \frac{1}{12} \) ft (.083 ft)).

Let \( V \) be the volume of the oil from tanker and \( r \) be the radius of the circle with a thickness of \( \frac{1}{10} \)" (oil outside tanker). Note that the volume of oil outside of tanker is the same as it is inside.

We know \( \frac{dV}{dt} = 150 \text{ gal/min} = \frac{150}{7.5} = 20 \text{ ft}^3/\text{min} \) and the relation of \( V \) and \( r \) is: \( V = \pi r^2 \left( \frac{1}{10(12)} \right) = \frac{\pi}{120} r^2 \) and the relation of \( \frac{dV}{dt} \) and \( \frac{dr}{dt} \) is: \( \frac{dV}{dt} = \frac{\pi}{120} (2r) = \frac{\pi}{60} r \frac{dr}{dt} \). Hence,

\[
\frac{dr}{dt} = \frac{60}{\pi r} \frac{dV}{dt}.
\]

When \( r = 500 \) ft,

\[
\frac{dr}{dt} = \frac{60}{\pi (500)} 20 = \frac{12}{5} \pi = 0.7694 \text{ ft/min}
\]

Example:

A 10-foot ladder leans against the side of a building. If the top of the ladder begins to slide down the wall at the rate of 2 ft/sec, how fast is the bottom of the ladder sliding away from the wall when the top of the ladder is 8 ft of the ground?

Let \( x(t) \) be the distance from the bottom of the ladder to the wall horizontally and \( y(t) \) be the distance from the top of the ladder to the floor vertically.

(1) Relation of \( x(t) \) and \( y(t) \) : \( x^2 + y^2 = 10^2 \)

(2) Relation of \( \frac{dx}{dt} \) and \( \frac{dy}{dt} \) : \( 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \).

(3) Know that \( \frac{dy}{dt} = -2 \) and \( y = 8 \), find \( \frac{dx}{dt} \).

\[
\frac{dx}{dt} = -\frac{2y}{2x} \frac{dy}{dt} = -\frac{8}{2 \times \frac{8}{\sqrt{10^2-8^2}} (-2)} = 2.67 \text{ ft/sec}
\]