4.2 - Sums and Sigma Notation

1. Sums and Sigma Notation

The sum of the first 10 integers:

$$1+2+3+4+5+6+7+8+9+10 = 5(10)+5 = 55$$

The sum of the first 100 integers:

$$1+2+3+4+\ldots+99+100 = 50(100)+50 = 5050$$

Let $n_1, \ldots, n_{10}$ be 10 positive integers. The sum of these 10 positive integers: $n_1 + \ldots + n_{10}$.

Sigma notation (**summation**):

$$\sum_{i=1}^{100} i = 1+2+3+4+5+\cdots+99+100$$

$$\sum_{i=-5}^{5} i^2 = (-5)^2+(-4)^2+(-3)^2+(-2)^2+(-1)^2+1+2^2+3^2+4^2+5^2$$

$$\sum_{i=5}^{20} \sqrt{i} = \sqrt{5}+\sqrt{6}+\sqrt{7}+\cdots+\sqrt{20}$$

$$\sum_{i=0}^{200} e^{-2i} = 1+e^{-2}+e^{-4}+\cdots+e^{-400}$$

$$\sum_{i=0}^{30} \cos(ix) = 1+\cos(x)+\cos(2x)+\cdots+\cos(30x)$$

Example: Write the expression in summation notation:

a. $1^2 + 2^2 + \ldots + 1000^2$ b. $\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{4000}$

c. $1 + e + e^2 + \ldots + e^{2006}$

a. $1^2 + 2^2 + \ldots + 1000^2 = \sum_{i=1}^{100} i^2$

b. $\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{4000} = \sum_{i=1}^{1000} \sqrt{i}$

c. $1 + e + e^2 + \ldots + e^{2006} = \sum_{i=0}^{2006} e^i$

2. **Properties of Summation**:

Let n be any positive integer and c is any constant. Then

(1) $\sum_{i=1}^{n}(n_i + m_i) = \sum_{i=1}^{n} n_i + \sum_{i=1}^{n} m_i$

(2) $\sum_{i=1}^{n} c = cn$

(3) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(4) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(5) $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

1.

Example: *Evaluate*

a. $\sum_{i=3}^{20}(i + i^2 - 1)$ b. $\sum_{i=1}^{50}(2i - 3)^2$

a. $\sum_{i=3}^{20}(i + i^2 - 1) = \sum_{i=3}^{20} i + \sum_{i=3}^{20} i^2 - \sum_{i=3}^{20}(1) - (1+2) - (1+2^2)$

$= \frac{20(21)}{2} + \frac{20(21)(41)}{6} - (20-2) - 3 - 5 = 3054$

b. $\sum_{i=1}^{50}(2i - 3)^2 = \sum_{i=1}^{50}(4i^2 - 12i + 9) = 4\sum_{i=1}^{50} i^2 - 12\sum_{i=1}^{50} i + 9\sum_{i=1}^{50} 1$

$= 4\frac{(50)(51)(101)}{6} - 12\frac{50(51)}{2} - 9(50) = 155\,950$

Example: *Compute* $\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{2i}{n}\right)^2 - 4\left(\frac{i}{n}\right) + 3\right]$ *in terms of* n *and then find the limit of the sum as* $n \to \infty$.

$$\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{2i}{n}\right)^2 - 4\left(\frac{i}{n}\right) + 3\right]$$

$$= \sum_{i=1}^{n} \frac{1}{n}\left[\frac{4i^2}{n^2} - 4\frac{i}{n} + 3\right] = \frac{4}{n^3}\sum_{i=1}^{n} i^2 - \frac{4}{n^2}\sum_{i=1}^{n} i + \frac{3}{n}\sum_{i=1}^{n}(1)$$

$$= \frac{4}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{4}{n^2}\frac{n(n+1)}{2} + \frac{3}{n}(n)$$

$$= \frac{2(n+1)(2n+1)}{3n^2} - 2\frac{n+1}{n} + 3$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{2i}{n}\right)^2 - 4\left(\frac{i}{n}\right) + 3\right]$$

$$= \lim_{n\to\infty}\left[\frac{2(n+1)(2n+1)}{3n^2} - 2\frac{n+1}{n} + 3\right] = \frac{4}{3} - 2 + 3 = \frac{7}{3}$$

Example: *Evaluate the sum a.* $\sum_{i=1}^{10}\sqrt{1+i^2}$ b. $\sum_{i=0}^{5}\cos(\frac{i\pi}{10})$ *by TI-83plus or TI-89.*

TI-83plus: sum(seq($\sqrt{(1+x^2)}, x, 1, 10$))

sum - LIST/MATH/sum

seq - LIST/OPS/seq

TI-89: $\sum(\sqrt{(1+x^2)}, x, 1, 10)$

$\sum$ using F3/$\sum$

a. $\sum_{i=1}^{10}\sqrt{1+i^2} = 56.3560332$

b. $\sum_{i=0}^{5}\cos(\frac{i\pi}{10}) = 3.65687576$ (sum(seq($\cos(x\pi/10), x, 0, 5$))