

Homework Assignment 16 - (12.7) - Solutions

Page 1008: : - turn in the problems with (*)

51-54: 52*, 54*

55-60: 58*

1-8: 2*, 4*, 6*

35-38: 35*, 38*

Extra points: 44(b), 46

51. False. $\nabla f(a, b)$ may not exist.

52. False. (a, b) could be a saddle point, local minimum or neither.

53. False. There can be saddle points between two local maxima.

54. True.

58. Local extreme point at $(0, 1)$, saddle points at $(-2, -1)$, $(2, -1)$

2. $f(x, y) = \cos^2 x + y^2$

$$\nabla f(x, y) = [-2 \cos x \sin x, 2y] = [-\sin(2x), 2y] = \vec{0} \Rightarrow \begin{cases} (1) \sin(2x) = 0 \\ (2) y = 0 \end{cases}$$

$$(1) \sin(2x) = 0 \text{ if } 2x = n\pi, x = \frac{n\pi}{2}, n = 0, \pm 1, \pm 2, \dots$$

Critical points:

$$\left\{ \left(\frac{n\pi}{2}, 0 \right), n = 0, \pm 1, \pm 2, \dots \right\}$$

Second Derivative Test:

$$f_{xx} = -2 \cos(2x), f_{yy} = 2, f_{xy} = 0, D(x, y) = -4 \cos(2x)$$

$$D\left(\frac{n\pi}{2}, 0\right) = -4 \cos(n\pi) = \begin{cases} 4 > 0 \text{ if } n = 2k + 1, k = 0, 1, \dots \\ -4 < 0 \text{ if } n = 2k, k = 0, 1, \dots \end{cases}$$

When $n = 2k + 1$ (odd numbers):

$$f_{xx}\left(\frac{(2k+1)\pi}{2}, 0\right) = -2 \cos((2k+1)\pi) = 2 > 0.$$

So, points $\left(\frac{(2k+1)\pi}{2}, 0\right)$ are local minima and $\left(\frac{2k\pi}{2}, 0\right) = (k\pi, 0)$ are saddle points.

4. $f(x, y) = 4xy - x^4 - y^4 + 4$

$$\nabla f(x, y) = [4y - 4x^3, 4x - 4y^3] = \vec{0} \Rightarrow \begin{cases} (1) 4y - 4x^3 = 0, & y = x^3 \rightarrow (2) \\ (2) 4x - 4y^3 = 0, & 4(x - x^9) = 4x(1 - x^8) = 0 \\ \text{Answers: } & x = 0, x = \pm 1 \end{cases}$$

Critical points: $(0, 0)$, $(1, 1)$ and $(-1, -1)$.

Second Derivative Test:

$$f_{xx} = -12x^2, \quad f_{xy} = 4, \quad f_{yy} = -12y^2, \quad D(x,y) = (-12x^2)(-12y^2) - 16 = 144x^2y^2 - 16$$

$$D(0,0) = 0 - 16 < 0, \text{ so } (0,0) \text{ is a saddle point.}$$

$$D(1,1) = 144 - 16 > 0, \quad f_{xx}(1,1) = -12 < 0, \text{ so } (1,1) \text{ is a local maximum.}$$

$$D(-1,-1) = 144 - 16 > 0, \quad f_{xx}(-1,-1) = -12 < 0, \text{ so } (-1,-1) \text{ is also a local maximum.}$$

6. $f(x,y) = 2x^2 + y^3 - x^2y - 3y$

$$\nabla f(x,y) = [4x - 2xy, 3y^2 - x^2 - 3] = \vec{0}$$

$$\left\{ \begin{array}{l} (1) 4x - 2xy = 2x(2 - y) = 0, \quad x = 0 \text{ or } y = 2 \\ (2) 3y^2 - x^2 - 3 = 0 \Rightarrow \begin{cases} \text{when } x = 0, 3(y^2 - 1) = 0, y = \pm 1 \\ \text{when } y = 2, 3(4) - x^2 - 3 = -x^2 + 9 = 0, x = \pm 3 \end{cases} \end{array} \right.$$

Critical points: $(0, 1)$, $(0, -1)$, $(3, 2)$ and $(-3, 2)$.

Second Derivative Test:

$$f_{xx} = 4 - 2y, \quad f_{xy} = -2x, \quad f_{yy} = 6y, \quad D(x,y) = 12(2 - y)y - 4x^2$$

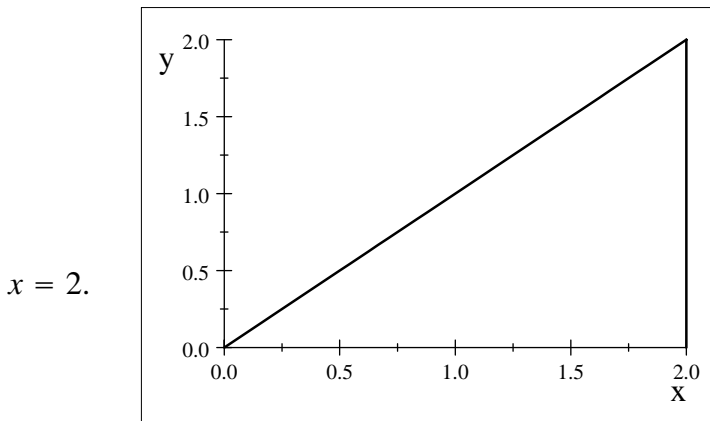
$$D(0,1) = 12 > 0, \quad f_{xx}(0,1) = 2 > 0, \text{ so } (0,1) \text{ is a local minimum}$$

$$D(0,-1) = 12(3)(-1) < 0, \quad (0, -1) \text{ is a saddle point}$$

$$D(3,2) = 0 - 36 < 0, \quad (3,2) \text{ is a saddle point}$$

$$D(-3,2) = 0 - 36 < 0, \quad (-3,2) \text{ is also a saddle point}$$

35. $f(x,y) = x^2 + 3y - 3xy$, $R : y = x$, $y = 0$, and



a. Find all critical points:

$$\nabla f(x,y) = [2x - 3y, 3 - 3x] = \vec{0} \Rightarrow \begin{cases} (1) 2x - 3y = 0, y = \frac{2x}{3} \\ (2) 3 - 3x = 3(1 - x) = 0, x = 1 \end{cases}$$

Critical point: $(1, \frac{2}{3})$. $f(1, \frac{2}{3}) = 1$

b. Find absolute extrema on the boundary. The boundary R consists of 3 curves:

$$C_1 : y = 0, 0 \leq x \leq 2; \quad C_2 : x = 2, 0 \leq y \leq 2; \quad C_3 : y = x, 0 \leq x \leq 2$$

i. $C_1 : g(x) = f(x,0) = x^2, 0 \leq x \leq 2$

Critical points:

$$g'(x) = 2x = 0, x = 0 \text{ is a boundary point}$$

Boundary points: (0,0), (2,0)

	$f(x,y)$
(0,0)	0
(2,0)	4

ii. $C_2 : h(y) = f(2,y) = 2^2 + 3y - 3(2)y = 4 + 3y - 6y = 4 - 3y, 0 \leq y \leq 2$

Critical points:

$$h'(y) = -3 \text{ - no critical point}$$

Boundary points: (2,0), (2,2)

	$f(x,y)$
(2,0)	4
(2,2)	-2

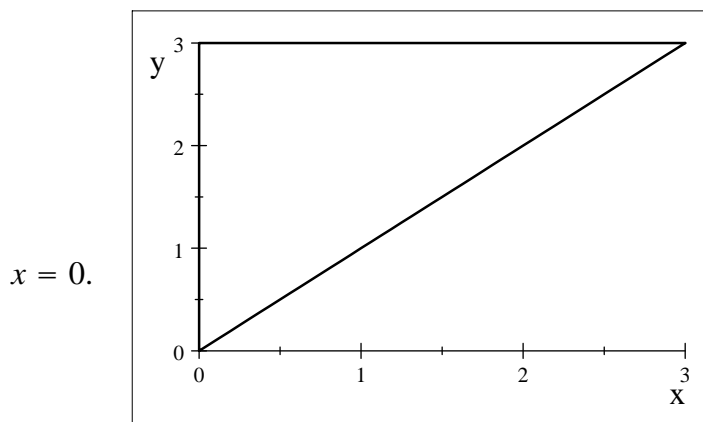
iii. $C_3 : g(x) = f(x,x) = x^2 + 3x - 3x^2 = -2x^2 + 3x = -x(2x - 3), 0 \leq x \leq 2$.
 g is a parabola and is concave down so g has a local maximum at $x = \frac{1}{2}$.

$$g(0) = 0, g(2) = -2, g\left(\frac{1}{2}\right) = -\left(\frac{1}{2}\right)\left(\frac{2}{2} - 3\right) = \frac{3}{4}$$

	$f(x,y)$
$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\frac{3}{4}$
(2,2)	-2

The absolute maximum value 4 is obtained at (2,0) and the absolute minimum -2 is obtained at (2,2).

38. $f(x,y) = x^2 + y^2 - 2x - 4y, R : y = x, y = 3, \text{ and}$



a. Find all critical points:

$$\nabla f(x,y) = [2x-2, 2y-4] = \vec{0} \Rightarrow \begin{cases} (1) 2x-2 = 2(x-1) = 0, & x = 1 \\ (2) 2y-4 = 2(y-2) = 0, & y = 2 \end{cases}$$

Critical point: $(1, 2)$. $f(1,2) = -5$

b. Find absolute extrema on the boundary. The boundary R consists of 3 curves:

$$C_1 : y = 3, 0 \leq x \leq 3; \quad C_2 : x = 0, 0 \leq y \leq 3; \quad C_3 : y = x, 0 \leq x \leq 3$$

i. $C_1 : g(x) = f(x,3) = x^2 + 3^2 - 2x - 4(3) = x^2 - 2x - 3, 0 \leq x \leq 3$

Critical point:

$$g'(x) = 2x - 2 = 0, x = 1, f(1,3) = -4$$

Boundary points: $(0,3), (3,-3)$

$$g(3) = 9 - 6 - 3 = 0, g(0) = -3$$

The absolute maximum value 0 is obtained at $(3,3)$ and the absolute minimum value -4 is obtained at $(1,3)$.

ii. $C_2 : h(y) = f(0,y) = y^2 - 4y, 0 \leq y \leq 3$

Critical point:

$$h'(y) = 2y - 4 = 0, y = 2, (0,2), f(0,2) = -4$$

Boundary points: $(0,0), (0,3)$

$$h(3) = 9 - 12 = -3, h(0) = 0$$

The absolute maximum value 0 is obtained at $(0,0)$ and the absolute minimum value -4 is obtained at $(0,2)$.

iii. $C_3 : g(x) = f(x,x) = x^2 + x^2 - 2x - 4x = 2x^2 - 6x, 0 \leq x \leq 3$.

Critical point:

$$g'(x) = 4x - 6 = 0, x = \frac{3}{2}, f\left(\frac{3}{2}, \frac{3}{2}\right) = g\left(\frac{3}{2}\right) = \frac{9}{2} - 9 = -\frac{9}{2}$$

Boundary points: $(0,0), (3,3)$

$$g(3) = 18 - 18 = 0, g(0) = 0$$

The absolute maximum value 0 is obtained at $(3,3)$ and $(0,0)$, and the absolute minimum value $-\frac{9}{2}$ is obtained at $\left(\frac{3}{2}, \frac{3}{2}\right)$.

Comparing values of $f(x,y)$ in a., b.i, b.ii, and b.iii, the absolute minimum value -5 is reached at $(1,2)$ and the absolute maximum value 0 is obtained at $(3,3)$ and $(0,0)$

46. $f(x,y) = 4xy - x^4 - y^4 + 4$, $(0,0)$ is a saddle point.

Let $x = 0$ and $g(y) = f(0,y) = -y^4 + 4$. g has a local maximum value 4 at $y = 0$. So, $(0,0)$ is a local minimum of the trace $g(y) = 4 - y^4$.

Let $y = kx$ and $g(x) = f(x,kx) = 4x(kx) - x^4 - (kx)^4 + 4 = 4kx^2 - x^4 - k^4x^4 + 4$.

$$g'(x) = 8kx - 4x^3 - 4k^4x^3, g'(0) = 0.$$

So, $x = 0$ is a critical point of $g(x)$. Check $g''(0)$:

$$g''(x) = 8k - 12x^2 - 12k^4x^2, g''(0) = 8k > 0 \text{ if and only if } k > 0.$$

Let $y = x$. Then $g(x) = f(x,x) = 4x^2 - 2x^4 + 4$. g has a local minimum at $(0,0)$.

