Homework Assignment 16 - (12.7) - Solutions

Page 1008: turn in the problems with (*)
51-54: 52*, 54*
55-60: 58*
1-8: 2*, 4*, 6*
35-38: 35*, 38*
Extra points: 44(b), 46

51. False. \( \nabla f(a, b) \) may not exist.

52. False. \((a, b)\) could be a saddle point, local minimum or neither.

53. False. There can be saddle points between two local maxima.

54. True.

58. Local extreme point at \((0, 1)\), saddle points at \((-2, -1), (2, -1)\)

2. \( f(x, y) = \cos^2 x + y^2 \)

\[
\nabla f(x, y) = \left[ -2 \cos x \sin x, \ 2y \right] = \left[ -\sin(2x), \ 2y \right] = \vec{0} \implies \begin{cases} (1) \ \sin(2x) = 0 \\ (2) \ y = 0 \end{cases}
\]

\( x = \frac{n \pi}{2}, \ n = 0, \pm 1, \pm 2, \ldots \)

Critical points:
\[
\left\{ \left( \frac{n \pi}{2}, \ 0 \right), \ n = 0, \pm 1, \pm 2, \ldots \right\}
\]

Second Derivative Test:
\[
f_{xx} = -2 \cos(2x), \ f_{yy} = 2, \ f_{xy} = 0, \quad D(x, y) = -4 \cos(2x)
\]

\[
D\left( \frac{n \pi}{2}, \ 0 \right) = -4 \cos(n \pi) = \begin{cases} 4 > 0 \text{ if } n = 2k + 1, \ k = 0, 1, \ldots \\ -4 < 0 \text{ if } n = 2k, \ k = 0, 1, \ldots \end{cases}
\]

When \( n = 2k + 1 \) (odd numbers):

\[
f_{xx}\left( \frac{(2k + 1) \pi}{2}, \ 0 \right) = -2 \cos((2k + 1) \pi) = 2 > 0
\]

So, points \( \left( \frac{(2k + 1) \pi}{2}, \ 0 \right) \) are local minima and \( \left( \frac{2k \pi}{2}, \ 0 \right) = (k \pi, \ 0) \) are saddle points.

4. \( f(x, y) = 4xy - x^4 - y^4 + 4 \)

\[
\nabla f(x, y) = \left[ 4y - 4x^3, \ 4x - 4y^3 \right] = \vec{0} \implies \begin{cases} (1) \ 4y - 4x^3 = 0, \ y = x^3 \rightarrow (2) \\ (2) \ 4x - 4y^3 = 0, \ 4(x - x^9) = 4(x(1 - x^8)) = 0 \end{cases}
\]

Answers: \( x = 0, \ x = \pm 1 \)

Critical points: \((0, 0), (1, 1)\) and \((-1, -1)\).
Second Derivative Test:
\[ f_{xx} = -12x^2, \quad f_{xy} = -12y^2, \quad D(x,y) = (-12x^2)(-12y^2) - 16 = 144x^2y^2 - 16 \]
\[ D(0,0) = 0 - 16 < 0, \text{ so, } (0,0) \text{ is a saddle point.} \]
\[ D(1,1) = 144 - 16 > 0, \quad f_{xx}(1,1) = -12 < 0, \text{ so } (1,1) \text{ is a local maximum.} \]
\[ D(-1,-1) = 144 - 16 > 0, \quad f_{xx}(-1,-1) = -12 < 0, \text{ so } (-1,-1) \text{ is also a local maximum.} \]

6. \( f(x,y) = 2x^2 + y^3 - x^2y - 3y \)
\[ \nabla f(x,y) = \left[ 4x - 2xy, \ 3y^2 - x^2 - 3 \right] = \mathbf{0} \]
\[ (1) 4x - 2xy = 2x(2 - y) = 0, \ x = 0 \text{ or } y = 2 \]
\[ (2) 3y^2 - x^2 - 3 = 0 \Rightarrow \begin{cases} 
\text{when } x = 0, & 3(y^2 - 1) = 0, \ y = \pm 1 \\
\text{when } y = 2, & 3(4) - x^2 - 3 = -x^2 + 9 = 0, \ x = \pm 3 
\end{cases} \]
Critical points: \( (0, 1), (0, -1), (3, 2) \) and \( (-3, 2) \).
Second Derivative Test:
\[ f_{xx} = 4 - 2y, \quad f_{xy} = -2x, \quad f_{yy} = 6y, \quad D(x,y) = 12(2 - y)y - 4x^2 \]
\[ D(0,1) = 12 > 0, f_{xx}(0,1) = 2 > 0, \text{ so } (0,1) \text{ is a local minimum} \]
\[ D(0,-1) = 12(3)(-1) < 0, \ (0, -1) \text{ is a saddle point} \]
\[ D(3,2) = 0 - 36 < 0, \ (3,2) \text{ is a saddle point} \]
\[ D(-3,2) = 0 - 36 < 0, \ (-3,2) \text{ is also a saddle point} \]

35. \( f(x,y) = x^2 + 3y - 3xy, \ R : y = x, \ y = 0, \) and \( x = 2. \)

a. Find all critical points:
\[ \nabla f(x,y) = \left[ 2x - 3y, \ 3 - 3x \right] = \mathbf{0} \Rightarrow \begin{cases} 
(1) \ 2x - 3y = 0, \ y = \frac{2x}{3} \\
(2) \ 3 - 3x = 3(1 - x) = 0, \ x = 1 
\end{cases} \]
Critical point: \( \left( 1, \frac{2}{3} \right) \). \( f\left( 1, \frac{2}{3} \right) = 1 \)

b. Find absolute extrema on the boundary. The boundary \( R \) consists of 3 curves:
\[ C_1 : y = 0, \ 0 \leq x \leq 2; \quad C_2 : x = 2, \ 0 \leq y \leq 2; \quad C_3 : y = x, \ 0 \leq x \leq 2 \]

i. \( C_1 : g(x) = f(x,0) = x^2, \ 0 \leq x \leq 2 \)

**Critical points:**

\[ g'(x) = 2x = 0, \ x = 0 \text{ is a boundary point} \]

**Boundary points:** (0, 0), (2, 0)

\[
\begin{array}{c|c}
(0,0) & 0 \\
(2,0) & 4 \\
\end{array}
\]

ii. \( C_2 : h(y) = f(2,y) = 2^2 + 3y - 3(2)y = 4 + 3y - 6y = 4 - 3y, \ 0 \leq y \leq 2 \)

**Critical points:**

\[ h'(y) = -3 \text{ - no critical point} \]

**Boundary points:** (2, 0), (2, 2)

\[
\begin{array}{c|c}
(2,0) & 4 \\
(2,2) & -2 \\
\end{array}
\]

iii. \( C_3 : g(x) = f(x,x) = x^2 + 3x - 3x^2 = -2x^2 + 3x = -x(2x-3), \ 0 \leq x \leq 2 \).

g is a parabola and is concave down so \( g \) has a local maximum at \( x = \frac{1}{3} \).

\[ g(0) = 0, \ g(2) = -2, \ g\left(\frac{1}{3}\right) = -\left(\frac{1}{3}\right)\left(\frac{2}{3} - 3\right) = \frac{7}{9} \]

\[
\begin{array}{c|c}
\left(\frac{1}{3},\frac{1}{3}\right) & \frac{7}{9} \\
(2,2) & -2 \\
\end{array}
\]

The absolute maximum value 4 is obtained at (2, 0) and the absolute minimum \(-2\) is obtained at (2, 2).

\[ \text{38. } f(x,y) = x^2 + y^2 - 2x - 4y, \ R : y = x, \ y = 3, \ \text{and} \]

\[
\begin{array}{c|c|c}
\hline
\text{x} & \text{y} & \text{x} \\
\hline
0 & 0 & 0 \\
1 & 1 & 1 \\
2 & 2 & 2 \\
3 & 3 & 3 \\
\hline
\end{array}
\]

\[ x = 0. \]

\[ \text{a. Find all critical points:} \]
\[ \nabla f(x,y) = [2x - 2, 2y - 4] = \vec{0} \Rightarrow \begin{cases} 
(1) \ 2x - 2 = 2(x - 1) = 0, \ x = 1 \\
(2) \ 2y - 4 = 2(y - 2) = 0, \ y = 2 
\end{cases} \]

**Critical point:** \((1, 2)\). \(f(1,2) = -5\)

b. Find absolute extrema on the boundary. The boundary \(R\) consists of 3 curves:
\[
C_1 : y = 3, \ 0 \leq x \leq 3; \quad C_2 : x = 0, \ 0 \leq y \leq 3; \quad C_3 : y = x, \ 0 \leq x \leq 3
\]
i. \(C_1 : g(x) = f(x,3) = x^2 + 3^2 - 2x - 4(3) = x^2 - 2x - 3, \ 0 \leq x \leq 3\)

**Critical point:**
\[ g'(x) = 2x - 2 = 0, \ x = 1, \ f(1,3) = -4 \]

**Boundary points:** \((0,3), \ (3,-3)\)

\[ g(3) = 9 - 6 - 3 = 0, \ g(0) = -3 \]

The absolute maximum value 0 is obtained at \((3,3)\) and the absolute minimum value –4 is obtained at \((1,3)\).

ii. \(C_2 : h(y) = f(0,y) = y^2 - 4y, \ 0 \leq y \leq 3\)

**Critical point:**
\[ h'(y) = 2y - 4 = 0, \ y = 2, \ (0,2), \ f(0,2) = -4 \]

**Boundary points:** \((0,0), \ (0,3)\)

\[ h(3) = 9 - 12 = -3, \ h(0) = 0 \]

The absolute maximum value 0 is obtained at \((0,0)\) and the absolute minimum value –4 is obtained at \((0,2)\).

iii. \(C_3 : g(x) = f(x,x) = x^2 + x^2 - 2x - 4x = 2x^2 - 6x, \ 0 \leq x \leq 3\).

**Critical point:**
\[ g'(x) = 4x - 6 = 0, \ x = \frac{3}{2}, \ f\left(\frac{3}{2}, \frac{3}{2}\right) = g\left(\frac{3}{2}\right) = \frac{9}{2} - 9 = -\frac{9}{2} \]

**Boundary points:** \((0,0), \ (3,3)\)

\[ g(3) = 18 - 18 = 0, \ g(0) = 0 \]

The absolute maximum value 0 is obtained at \((3,3)\) and \((0,0)\), and the absolute minimum value \(-\frac{9}{2}\) is obtained at \(\left(\frac{3}{2}, \frac{3}{2}\right)\).

Comparing values of \(f(x,y)\) in a., b.i, b.ii, and b.iii, the absolute minimum value –5 is reached at \((1,2)\) and the absolute maximum value 0 is obtained at \((3,3)\) and \((0,0)\)

46. \(f(x,y) = 4xy - x^4 - y^4 + 4, \ (0,0)\) is a saddle point.

Let \(x = 0\) and \(g(y) = f(0,y) = -y^4 + 4\). \(g\) has a local maximum value 4 at \(y = 0\). So, \((0,0)\) is a local minimum of the trace \(g(y) = 4 - y^4\).

Let \(y = kx\) and \(g(x) = f(x,kx) = 4x(kx) - x^4 - (kx)^4 + 4 = 4kx^2 - x^4 - k^4x^4 + 4\).

\[ g'(x) = 8kx - 4x^3 - 4k^4x^3, \ g'(0) = 0. \]

So, \(x = 0\) is a critical point of \(g(x)\). Check \(g''(0)\):

\[ g''(x) = 8k - 12x^2 - 12k^4x^2, \ g''(0) = 8k > 0 \text{ if and only if } k > 0. \]

Let \(y = x\). Then \(g(x) = f(x,x) = 4x^2 - 2x^4 + 4\). \(g\) has a local minimum at \((0,0)\).