Vectors In The Plane - 10.1

Questions:

- What is a vector in the plane? What is the difference between a vector and a line segment?
- How is a vector in the plane determined? What are forms of a vector in the plane?

1. **Vectors**

   A vector $\vec{\mathbf{u}}$ in the plane is a **directed line segment** from an initial point $P(x_1,\ y_1)$ to a terminal point $Q(x_2,\ y_2)$, denoted by $\langle u_1, u_2\rangle$ where $u_1 = x_2 - x_1$ and $u_2 = y_2 - y_1$. The **direction** of $\vec{\mathbf{u}}$ is $\overrightarrow{PQ}$ and the **magnitude** of $\vec{\mathbf{u}}$ is the length of $\overrightarrow{PQ}$ defined by

   $$\|\vec{\mathbf{u}}\| = \sqrt{u_1^2 + u_2^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$ (the **distance** between P and Q).

   A vector is called a **position vector** if the initial point is at the origin $(0,0)$. Note that a vector in the plane is represented by a **direction** with a **size**. Elements u_1 and u_2 are called the **first and second components** of $\vec{\mathbf{u}}$.

Example *Find the vector with initial point at $P(2,3)$ and terminal point $Q(3,-1)$ and the vector with initial point at Q and the terminal point P.*

$$\vec{\mathbf{u}} = \overrightarrow{PQ} = \langle 3-2, -1-3\rangle = \langle 1,-4\rangle, \quad \vec{\mathbf{v}} = \overrightarrow{QP} = \langle 2-3, 3-(-1)\rangle = \langle -1,4\rangle$$

2. **Vector Algebra**

   Let $\vec{\mathbf{u}} = \langle u_1,\ u_2\rangle$ and $\vec{\mathbf{v}} = \langle v_1,\ v_2\rangle$ be vectors in the plane, and c be a constant. Then

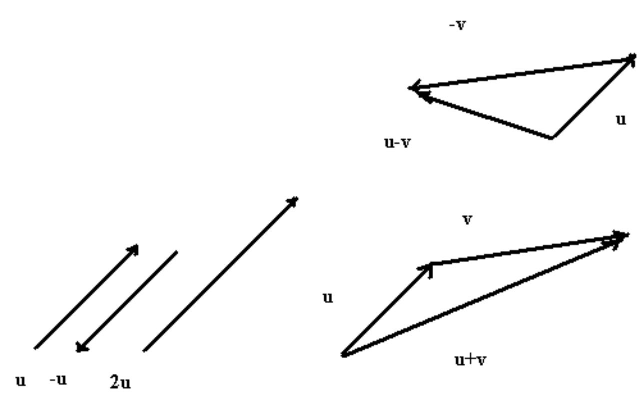
   a. **scalar multiplication**: $c\,\vec{\mathbf{u}} = \langle cu_1,\ cu_2\rangle$

   b. **vector addition**: $\vec{\mathbf{u}} \pm \vec{\mathbf{v}} = \langle u_1 \pm v_1,\ u_2 \pm v_2\rangle$

   Observe that

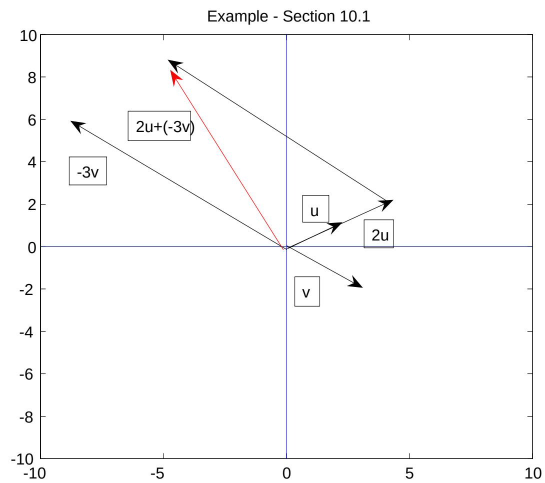
   $$\|c\,\vec{\mathbf{u}}\| = \sqrt{(cu_1)^2 + (cu_2)^2} = \sqrt{c^2u_1^2 + c^2u_2^2} = \sqrt{c^2(u_1^2 + u_2^2)} = |c|\sqrt{u_1^2 + u_2^2} = |c|\ \|\vec{\mathbf{u}}\|.$$

   $c\vec{\mathbf{u}}$ changes the magnitude of $\vec{\mathbf{u}}$ is $c \neq \pm 1$. When $c < 0$, $c\vec{\mathbf{u}}$ and $\vec{\mathbf{u}}$ have the opposite direction. Graphically, $\vec{\mathbf{u}} + \vec{\mathbf{v}}$ is a vector from the initial point of $\vec{\mathbf{u}}$ to the end point of $\vec{\mathbf{v}}$ after adding it to the end point of $\vec{\mathbf{u}}$. The following shows the graphs of $\vec{\mathbf{u}}$, $-\vec{\mathbf{u}}$, $2\vec{\mathbf{u}}$, $\vec{\mathbf{u}} + \vec{\mathbf{v}}$, and $\vec{\mathbf{u}} - \vec{\mathbf{v}}$:



Example *For vectors* $\vec{\mathbf{u}} = \langle 2, 1\rangle$ *and* $\vec{\mathbf{v}} = \langle 3, -2\rangle$, *compute* (i) $2\vec{\mathbf{u}} - 3\vec{\mathbf{v}}$ (ii) $||3\vec{\mathbf{u}} + \vec{\mathbf{v}}||$.

(i) $2\vec{\mathbf{u}} - 3\vec{\mathbf{v}} = \langle 4, 2\rangle - \langle 9, -6\rangle = \langle -5, 8\rangle$



(ii) $3\vec{\mathbf{u}} + \vec{\mathbf{v}} = \langle 6, 3\rangle + \langle 3, -2\rangle = \langle 9, 1\rangle$, $||3\vec{\mathbf{u}} + \vec{\mathbf{v}}|| = \sqrt{9^2 + 1^2} = \sqrt{10}$.

3. Special Vectors

- **Parallel vectors:** Two vectors are said to be **parallel** if they have the same or opposite direction. Algebraically, vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are parallel if there is a nonzero constant c such that $\vec{\mathbf{u}} = c\,\vec{\mathbf{v}}$, or
$$\frac{u_1}{v_1} = \frac{u_2}{v_2} = c.$$
- **Zero vector:** The **zero vector** $\vec{\mathbf{v}} = \langle 0, 0\rangle$ is the only vector with **zero length**.
- **Unit vector:** Any vector $\vec{\mathbf{u}}$ with $||\vec{\mathbf{u}}|| = 1$ is called a **unit vector**. A nonzero vector $\vec{\mathbf{v}}$ can be **normalized** as a unit vector:
$$\vec{\mathbf{u}} = \frac{1}{||\vec{\mathbf{v}}||}\,\vec{\mathbf{v}}.$$
- **Standard basis vectors:** $\vec{\mathbf{i}} = \langle 1, 0\rangle$ and $\vec{\mathbf{j}} = \langle 0, 1\rangle$ are called the **standard basis vectors**. Let $\vec{\mathbf{u}} = \langle u_1,\ u_2\rangle$. Then $\vec{\mathbf{u}} = u_1\vec{\mathbf{i}} + u_2\vec{\mathbf{j}}$.

Example *Determine whether or not the given pair vectors is* ***parallel***.

(i) $\vec{\mathbf{u}} = \langle 2, 3\rangle$, $\vec{v} = \langle 4, 5\rangle$ (ii) $\vec{\mathbf{u}} = \langle 2, -3\rangle$, $\vec{v} = \langle -4, 6\rangle$

Check if $\vec{\mathbf{u}} = c\,\vec{\mathbf{v}}$.

(i) $\frac{u_1}{v_1} = \frac{2}{4} = \frac{1}{2}$, $\frac{u_2}{v_2} = \frac{3}{5} \neq \frac{1}{2}$ so $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are not parallel.

(ii) $\frac{u_1}{v_1} = \frac{2}{-4} = -\frac{1}{2}$, $\frac{u_2}{v_2} = \frac{-3}{6} = -\frac{1}{2}$ so $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are parallel.

Example *Find a vector* $\vec{\mathbf{v}}$ *which is* ***parallel*** *to the vector* $\vec{\mathbf{u}} = \langle 1, -2\rangle$ *and has a* ***magnitude*** *of* 2.

We know that $\vec{\mathbf{v}} = c\,\vec{\mathbf{u}}$ for some constant c. Since
$||\vec{\mathbf{v}}|| = |c|||\vec{\mathbf{u}}|| = |c|\sqrt{1^2 + (-2)^2} = \sqrt{5}\,|c| = 2$, $|c| = \frac{2}{\sqrt{5}}$.

Hence, $c = \frac{2}{\sqrt{5}}$ or $c = -\frac{2}{\sqrt{5}}$ and $\vec{\mathbf{v}} = \frac{2}{\sqrt{5}}\langle 1,-2\rangle$ or $\vec{\mathbf{v}} = -\frac{2}{\sqrt{5}}\langle 1,-2\rangle$.

Example *Find a **unit vector** in the same direction as* $\vec{\mathbf{u}} = \langle 3,-4\rangle$.

$$\|\vec{\mathbf{u}}\| = \sqrt{3^2+(-4)^2} = 5, \quad \vec{\mathbf{v}} = \frac{1}{\|\vec{\mathbf{u}}\|}\vec{\mathbf{u}} = \frac{1}{5}\langle 3,-4\rangle = \left\langle \frac{3}{5},-\frac{4}{5}\right\rangle.$$

4. Polar Form of a Vector

It is often convenient to express a vector explicitly in terms of its **direction** in the plane and its **magnitude**. Let $\vec{\mathbf{u}} = \langle x,y\rangle$. Its direction can be expressed by $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, the angle between the positive x-axis and $\vec{\mathbf{u}}$. $\vec{\mathbf{u}}$ is said to be in **polar form** if

$$\vec{\mathbf{u}} = r(\cos\theta,\sin\theta) = (r\cos\theta, r\sin\theta), \quad \text{where } r = \|\vec{\mathbf{u}}\| = \sqrt{x^2+y^2} \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

Note that $\tan\theta = \frac{y}{x}$ implies $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$ and $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$. Then

$$\vec{\mathbf{u}} = \langle x,y\rangle \overset{\text{polar form}}{=} \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}}\right)$$

Example *Write the given vector* $\vec{\mathbf{u}}$ *in **polar form**.*

$$(i)\ \vec{\mathbf{u}} = \langle -1,2\rangle \qquad (ii)\ \vec{\mathbf{v}} = 2\vec{i}-2\vec{j}$$

(i) $r = \|\vec{\mathbf{u}}\| = \sqrt{(-1)^2+2^2} = \sqrt{5}$, $\theta = \tan^{-1}\left(\frac{2}{-1}\right) = -\arctan 2 = -1.1071487$ radians

$$\vec{\mathbf{u}} = \sqrt{5}\left\langle -\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right\rangle$$

(ii) $r = \|\vec{\mathbf{v}}\| = \sqrt{2^2+(-2)^2} = \sqrt{8}$, $\theta = \tan^{-1}(-\frac{2}{2}) = -\frac{\pi}{4}$,

$$\vec{\mathbf{v}} = \sqrt{8}\left\langle \frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right\rangle = \sqrt{8}\left\langle \cos\left(-\frac{\pi}{4}\right),\ \sin\left(-\frac{\pi}{4}\right)\right\rangle$$