Vectors In Space - 10.2

Questions:
- What is the right-handed coordinate system?
- What is the distance formula between two points in space? What is the equation of a sphere with center \((a, b, c)\) and radius \(r\)?
- How is a vector in the plane determined?
- What does it mean if three or more points in space are said to be colinear?

1. The Right-handed Coordinate System
   We use the right-handed \((x, y, z)\) coordinate system:

   A point in the space can be represented by \((a, b, c)\) where \(a, b, \) and \(c\) are real numbers. The right-handed coordinate system consists of three axes: \(x\) –axis, \(y\) –axis, and \(z\) –axis, three planes: \(xy\) –plane, \(yz\) –plane, and \(xz\) –plane, and eight octants. We can describe the axes, the planes and octants as follows:
   \[
   \begin{align*}
   x\text{–axis: } y &= 0, \quad z = 0 \\
   y\text{–axis: } x &= 0, \quad z = 0 \\
   z\text{–axis: } x &= 0, \quad y = 0 \\
   xy\text{–plane: } z &= 0 \\
   yz\text{–plane: } x &= 0, \quad \text{first octant: } x > 0, \ y > 0, \ \text{and } z > 0 \\
   xz\text{–plane: } y &= 0
   \end{align*}
   \]

   Example  Describe the plane which is parallel to \(xy\) –plane and contains a point \((-1, 2, 3)\).

   Since it is parallel to \(xy\) –plane, the \(z\) coordinate of the point \((a, b, c)\) is a constant. Since the point \((-1, 2, 3)\) is in the plane and \(c = 3\), the \(z\) coordinate of every point in the plane is \(3\). So, the equation of this plane is: \(z = 3\).

2. Distance Between Two Points

1
The distance between two points \( P(x_1, y_1, z_1) \) and \( Q(x_2, y_2, z_2) \) is
\[ d\{(x_1, y_1, z_1), (x_2, y_2, z_2)\} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \]

**Example** Plot the points \( P(1, 2, 3), \ Q(3, -2, 4) \) and \( R(-1, 3, -2) \). Find the distance between \( Q \) and \( R \).

**Example** Find the equation of a sphere of radius \( r \) centered at the point \( (x_0, y_0, z_0) \).

Pick a point on the sphere, say \((x, y, z)\). We know the distance from this point \((x, y, z)\) to the center \((0, 0, 0)\) is \( r \). So
\[ d\{(x_0, y_0, z_0), (x, y, z)\} = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = r \]
and the equation of the sphere is:
\[ (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2. \]

**Example** Find an equation of the sphere with radius \( r = 2 \) and center at \((-1, 2, -3)\).

The equation is: \((x + 1)^2 + (y - 2)^2 + (z + 3)^2 = 4.\)

**Example** Find the radius and the center of the sphere if we know its equation is
\[ x^2 - 2x + y^2 + y + z^2 = 1. \]

Complete the squares for \( x \) and \( y \):
\[ x^2 - 2x + 1 - 1 + y^2 + 2\left(\frac{1}{2}\right)y + \left(\frac{1}{2}\right)^2 = 1 \]
\[ (x - 1)^2 + \left(y + \frac{1}{2}\right)^2 + z^2 = 1 + 1 + \frac{1}{4} = \frac{9}{4} \]
\[ r = \frac{3}{2}, \text{ center is at } (1, -\frac{1}{2}, 0). \]

3. **Vectors in \( \mathbb{R}^3 \)**

\( \mathbb{R}^3 \) is the notation for the three-dimensional Euclidean space. A vector \( \mathbf{u} \) in \( \mathbb{R}^3 \) is represented by any directed line segment with the appropriate magnitude and direction. The position vector (its initial
point is the origin) \( \overrightarrow{v} \) with terminal point \( Q(v_1, v_2, v_3) \) is denoted by \( \langle v_1, v_2, v_3 \rangle \). The magnitude of the position vector \( \overrightarrow{v} \) is

\[
|\overrightarrow{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}.
\]

Denote the set of all 3-dimensional position vectors by

\[
V_3 = \{ \langle x, y, z \rangle; \ x, y, z \text{ in } \mathbb{R} \}.
\]

4. Vector Algebra

a. scalar multiplication: \( c \overrightarrow{u} = \langle cu_1, cu_2, cu_3 \rangle \)

b. vector addition: \( \overrightarrow{u} \pm \overrightarrow{v} = \langle u_1 \pm v_1, u_2 \pm v_2, u_3 \pm v_3 \rangle \)

Example For vectors \( \overrightarrow{u} = \langle 2, 1, -3 \rangle \) and \( \overrightarrow{v} = \langle 3, -2, 5 \rangle \), compute (i) \( 2\overrightarrow{u} - 3\overrightarrow{v} \) (ii) \( ||3\overrightarrow{u} + \overrightarrow{v}|| \).

(i) \( 2\overrightarrow{u} - 3\overrightarrow{v} = \langle 4, 2, -6 \rangle - \langle 9, -6, 15 \rangle = \langle -5, 8, 9 \rangle \)

(ii) \( 3\overrightarrow{u} + \overrightarrow{v} = \langle 6, 3, -9 \rangle + \langle 3, -2, 5 \rangle = \langle 9, 1, -4 \rangle \)

\[
||3\overrightarrow{u} + \overrightarrow{v}|| = \sqrt{9^2 + 1^2 + (-4)^2} = 7\sqrt{2}.
\]

5. Special Vectors

Parallel vectors: Two vectors are said to be parallel if they have the same or opposite direction, i.e.,

\[
\overrightarrow{u} = c\overrightarrow{v}
\]

that is \( \frac{u_1}{v_1} = \frac{u_2}{v_2} = \frac{u_3}{v_3} = c \).

Zero vector: The zero vector \( \overrightarrow{0} = \langle 0, 0, 0 \rangle \) is the only vector with zero length.

Unit vector: Any vector \( \overrightarrow{u} \) with \(|\overrightarrow{u}| = 1\) is called a unit vector. A nonzero vector \( \overrightarrow{v} \) can be normalized as a unit vector:

\[
\frac{\overrightarrow{u}}{|\overrightarrow{v}|} = \overrightarrow{v}.
\]

Standard basis vectors: \( \overrightarrow{i} = \langle 1, 0, 0 \rangle \), \( \overrightarrow{j} = \langle 0, 1, 0 \rangle \) and \( \overrightarrow{k} = \langle 0, 0, 1 \rangle \) are called the standard basis vectors. Let \( \overrightarrow{u} = \langle u_1, u_2, u_3 \rangle \). Then \( \overrightarrow{u} = u_1\overrightarrow{i} + u_2\overrightarrow{j} + u_3\overrightarrow{k} \).

Example Find the vector which is parallel to the vector \( \overrightarrow{u} = \langle 1, -2, -3 \rangle \) and has a magnitude of \( 2 \).

Let \( \overrightarrow{v} \) be the vector. Then \( \overrightarrow{v} = c\overrightarrow{u} \). Since

\[
|\overrightarrow{v}| = |c||\overrightarrow{u}| = |c|\sqrt{1^2 + (-2)^2 + (-3)^2} = \sqrt{14}|c| = 2, \ |c| = \frac{2}{\sqrt{14}}.
\]

So, \( c = \frac{2}{\sqrt{14}} \) or \( c = -\frac{2}{\sqrt{14}} \).

Example Find a unit vector in the same direction as \( \overrightarrow{u} = \langle 3, -4, 1 \rangle \).

\[
|\overrightarrow{u}| = \sqrt{3^2 + (-4)^2 + 1} = \sqrt{26}, \ \overrightarrow{v} = \frac{1}{|\overrightarrow{u}|} \overrightarrow{u} = \frac{1}{\sqrt{26}} \langle 3, -4, 1 \rangle = \langle \frac{3}{5}, \frac{-4}{5}, \frac{1}{\sqrt{26}} \rangle.
\]
**Example**  Find the displacement vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$. Determine whether the points $P(2, 3, 1)$, $Q(0, 4, 2)$ and $R(4, 1, 4)$ are **colinear** (on the same line).

Displacement vectors:

\[
\overrightarrow{PQ} = \langle -2, 1, 1 \rangle = \vec{u}, \quad \overrightarrow{QR} = \langle 4, -3, 2 \rangle = \vec{v}.
\]

\[
\frac{u_1}{v_1} = \frac{-2}{4} = -\frac{1}{2}, \quad \frac{u_2}{v_2} = \frac{1}{-3} \neq -\frac{1}{2}.
\]

Since $\overrightarrow{PQ} \neq c \overrightarrow{QR}$, these three points are not colinear.