

## The Dot Product - (10.3)

### Questions:

- What is the definition of the dot product of two vectors? Is it a scalar or vector?
- What can you say about the dot product of two vectors (positive, negative or zero) if these two vectors are orthogonal, along the same direction or in the opposite direction?
- What is the projection vector of  $\vec{u}$  on  $\vec{v}$ ? What is the magnitude of this projection vector?

### 1. Dot Product

The **dot product** of two vectors  $\vec{u} = \langle u_1, u_2, u_3 \rangle$  and  $\vec{v} = \langle v_1, v_2, v_3 \rangle$  is defined by

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

The **dot product** of two vectors  $\vec{u} = \langle u_1, u_2 \rangle$  and  $\vec{v} = \langle v_1, v_2 \rangle$  is defined by

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2.$$

Notice that the dot product of two vectors is a scalar (instead of a vector).

**Example** Compute the dot product  $\vec{u} \cdot \vec{v}$  and  $\vec{v} \cdot \vec{w}$  where  $\vec{u} = \langle -1, 2, 3 \rangle$ ,  $\vec{v} = \langle 2, -1, 1 \rangle$  and  $\vec{w} = \langle 3, 2, -4 \rangle$ .

$$\vec{u} \cdot \vec{v} = \langle -1, 2, 3 \rangle \cdot \langle 2, -1, 1 \rangle = (-1)(2) + (2)(-1) + (3)(1) = -1$$

$$\vec{v} \cdot \vec{w} = \langle 2, -1, 1 \rangle \cdot \langle 3, 2, -4 \rangle = 2(3) + (-1)(2) + (1)(-4) = 0.$$

Notice that  $\vec{v} \cdot \vec{w} = 0$  but  $\vec{v} \neq \vec{0}$  and  $\vec{w} \neq \vec{0}$ .

### 2. Properties of the Dot Product

Let  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  be vectors and  $c$  be a constant. Then

a.  $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

b.  $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

c.  $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$

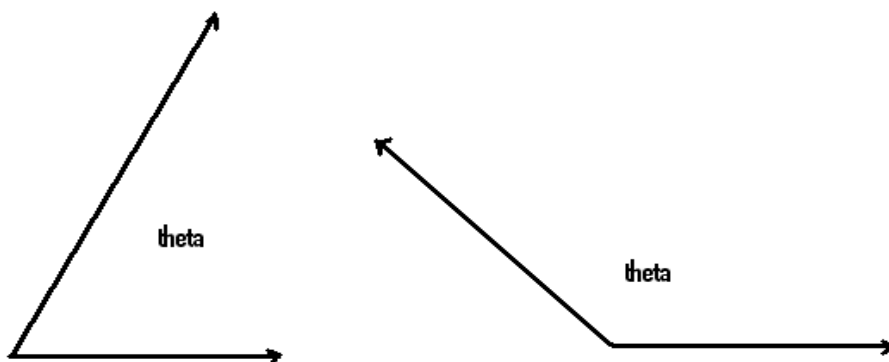
d.  $\vec{0} \cdot \vec{u} = 0$

e.  $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$

$$\vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + u_3^2 = \|\vec{u}\|^2$$

### 3. The Angle Between Two Vectors

Let  $\vec{u}$  and  $\vec{v}$  be two nonzero vectors. The **angle** between  $\vec{u}$  and  $\vec{v}$  in the plane obtained by  $\vec{u}$  and  $\vec{v}$  is defined by the **smaller angle**  $\theta$  between them where  $0 \leq \theta \leq \pi$ . For example,



Observe that

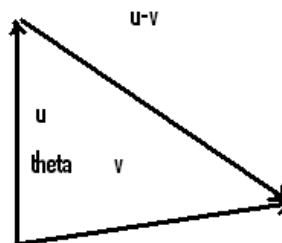
- a. If two vectors are in the **same direction**, then  $\theta = 0$ .
- b. If two vectors are in an **opposite direction**, then  $\theta = \pi$ .
- c. If two vectors are **orthogonal** or **perpendicular**, then  $\theta = \frac{\pi}{2}$ .

We consider the **zero vector** is orthogonal to **every vector**.

**Theorem** Let  $\theta$  be the *angle* between nonzero vectors  $\vec{u}$  and  $\vec{v}$ . Then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta.$$

**Proof** From the triangle formed by  $\vec{u}$ ,  $\vec{v}$  and  $\vec{u} - \vec{v}$ ,



,by the Law

of Cosines we have

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\| \|\vec{v}\| \cos \theta.$$

Observe that

$$\begin{aligned} \|\vec{u} - \vec{v}\|^2 &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} - \vec{v} \cdot \vec{u} - \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} \\ &= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u} \cdot \vec{v} \quad (\text{since } \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}). \end{aligned}$$

So we have

$$\|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\| \|\vec{v}\| \cos \theta = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u} \cdot \vec{v},$$

that shows

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta.$$

Notice that for nonzero vectors  $\vec{u}$  and  $\vec{v}$ ,  $\vec{u} \cdot \vec{v} = 0$  if and only if  $\cos \theta = 0$  or  $\theta = \frac{\pi}{2}$ . Hence,  $\vec{u}$  and  $\vec{v}$  are orthogonal if and only if  $\vec{u} \cdot \vec{v} = 0$ . From this theorem, we can find  $\theta$  as follows.

$$\theta = \cos^{-1} \left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right), \quad 0 \leq \theta \leq \pi.$$

**Example** (1) Find the angle between the vector  $\vec{u} = \langle 2, 1, -3 \rangle$  and  $\vec{v} = \langle 1, 5, 6 \rangle$  and determine whether or not they are orthogonal. (2) Find a vector  $\vec{w}$  perpendicular to  $\vec{u}$ .

$$(1) \vec{u} \cdot \vec{v} = 2 + 5 - 18 = -11, \quad \|\vec{u}\| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{14}, \quad \|\vec{v}\| = \sqrt{1^2 + 5^2 + 6^2} = \sqrt{62}$$

$$\theta = \cos^{-1}\left(\frac{-11}{\sqrt{14}\sqrt{62}}\right) = 1.953429.$$

Since  $\vec{u} \cdot \vec{v} \neq 0$ ,  $\vec{u}$  and  $\vec{v}$  are not orthogonal.

(2) Let  $\vec{w} = \langle a, b, c \rangle$ . Find  $a, b, c$  so that  $\vec{w} \cdot \vec{u} = 2a + b - 3c = 0$ . Let  $a = 1$ ,  $b = 1$  and  $c = 1$ . Then

$$\vec{w} = \langle 1, 1, 1 \rangle.$$

**Theorem (Cauchy-Schwartz Inequality)** For any vectors  $\vec{u}$  and  $\vec{v}$ ,

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$$

Since  $-1 \leq \cos \theta \leq 1$ ,

$$|\vec{u} \cdot \vec{v}| = \|\vec{u}\| \|\vec{v}\| \cos \theta \leq \|\vec{u}\| \|\vec{v}\|.$$

**Theorem (The Triangle Inequality)** For any vectors  $\vec{u}$  and  $\vec{v}$ ,

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|.$$

**Proof** Since  $\vec{u} \cdot \vec{v} \leq |\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$ ,

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2 \\ &\leq \|\vec{u}\|^2 + 2\|\vec{u}\| \|\vec{v}\| + \|\vec{v}\|^2 = (\|\vec{u}\| + \|\vec{v}\|)^2. \end{aligned}$$

Hence,

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|.$$

**Example** Let  $\vec{u} = \langle 2, 1, -3 \rangle$  and  $\vec{v} = \langle -4, 2, 1 \rangle$ . Verify the Cauchy-Schwartz inequality and the triangle inequality.

$$\vec{u} \cdot \vec{v} = (-8) + 2 + (-3) = -9, \quad \|\vec{u}\| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{14}, \quad \|\vec{v}\| = \sqrt{(-4)^2 + 2^2 + 1^2} = \sqrt{21}$$

Verify the Cauchy-Schwartz inequality:  $|\vec{u} \cdot \vec{v}| = 9 \leq \sqrt{14} \sqrt{21} = 17.14643$

$$\vec{u} + \vec{v} = \langle -2, 3, -2 \rangle, \quad \|\vec{u} + \vec{v}\| = \sqrt{(-2)^2 + 3^2 + (-2)^2} = \sqrt{17}$$

Verify the triangle inequality:  $\|\vec{u} + \vec{v}\| = \sqrt{17} = 4.123106 \leq \sqrt{14} + \sqrt{21} = 8.324233$ .

#### 4. Components and Projections

Let  $\vec{u}$  and  $\vec{v}$  be two nonzero vectors and  $\theta$  be the angle between them. Then the scalar

$$\text{comp}_{\vec{v}} \vec{u} = \|\vec{u}\| \cos \theta$$

is called the **component** of  $\vec{u}$  along  $\vec{v}$ . Notice that

$$\|\vec{u}\| \cos \theta = \frac{\|\vec{u}\| \|\vec{v}\| \cos \theta}{\|\vec{v}\|} = \frac{1}{\|\vec{v}\|} \vec{u} \cdot \vec{v}.$$

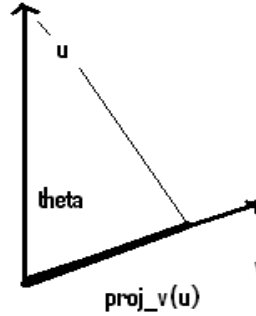
So, we can compute  $\text{comp}_{\vec{v}} \vec{u}$  **without** computing the **angle** between  $\vec{u}$  and  $\vec{v}$  first and by

$$\text{comp}_{\vec{v}} \vec{u} = \frac{1}{\|\vec{v}\|} \vec{u} \cdot \vec{v}.$$

The vector

$$\text{proj}_{\vec{v}} \vec{u} = (\text{comp}_{\vec{v}} \vec{u}) \left( \frac{\vec{v}}{\|\vec{v}\|} \right)$$

is called the **projection** of  $\vec{u}$  onto  $\vec{v}$ .



Notice that  $\frac{\vec{v}}{\|\vec{v}\|}$  is a **unit** vector in the same direction of  $\vec{v}$  and

$$\text{proj}_{\vec{v}} \vec{u} = \text{comp}_{\vec{v}} \vec{u} \left( \frac{\vec{v}}{\|\vec{v}\|} \right) = \frac{1}{\|\vec{v}\|} \vec{u} \cdot \vec{v} \left( \frac{\vec{v}}{\|\vec{v}\|} \right) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}.$$

Let  $\vec{w} = \vec{u} - \text{proj}_{\vec{v}} \vec{u}$ . Observe that

$$\vec{w} \cdot \vec{v} = \left( \vec{u} - \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} \right) \cdot \vec{v} = \vec{u} \cdot \vec{v} - \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} (\vec{v} \cdot \vec{v}) = \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{v} = 0$$

Note that  $\|\text{proj}_{\vec{v}} \vec{u}\| = |\text{comp}_{\vec{v}} \vec{u}|$ .

**Example** Let  $\vec{u} = \langle 1, -2, 3 \rangle$  and  $\vec{v} = \langle -2, 1, -3 \rangle$ . Find the component of  $\vec{u}$  along  $\vec{v}$  and the projection of  $\vec{u}$  along  $\vec{v}$ . Find a vector  $\vec{w}$  which is perpendicular to  $\vec{v}$ .

$$\vec{u} \cdot \vec{v} = \langle 1, -2, 3 \rangle \cdot \langle -2, 1, -3 \rangle = (-2) + (-2) + (-9) = -13$$

$$\|\vec{v}\| = \sqrt{(-2)^2 + 1 + (-3)^2} = \sqrt{14}$$

$$\text{comp}_{\vec{v}} \vec{u} = \frac{1}{\sqrt{14}} (-13) = -\frac{13}{\sqrt{14}}, \quad \text{proj}_{\vec{v}} \vec{u} = \frac{(-13)}{14} \langle -2, 1, -3 \rangle = \left\langle \frac{13}{7}, -\frac{13}{14}, \frac{39}{14} \right\rangle$$

$$\vec{w} = \langle 1, -2, 3 \rangle - \left\langle \frac{13}{7}, -\frac{13}{14}, \frac{39}{14} \right\rangle = \left\langle -\frac{6}{7}, -\frac{15}{14}, \frac{3}{14} \right\rangle.$$

Check if  $\vec{w} \cdot \vec{v} = 0$  :

$$\vec{w} \cdot \vec{v} = \left\langle -\frac{6}{7}, -\frac{15}{14}, \frac{3}{14} \right\rangle \cdot \langle -2, 1, -3 \rangle = \frac{24}{14} - \frac{15}{14} - \frac{9}{14} = 0$$

**Example** A force of 40 pounds is exerted in the direction of the handle of the wagon. If the handle makes an angle of  $\frac{\pi}{4}$  with the horizontal and the wagon is pulled along a flat surface for 1 mile (5280 feet), find the work done.

The work done can be measured by the product of the distance the wagon is pulled and the component of the force in the direction of the handle of the wagon along the horizontal direction where the wagon

is pulled.

$$W = 40 \cos\left(\frac{\pi}{4}\right)(5280) = 149341 \text{ foot-pounds.}$$