The Cross Product - (10.4)

Questions:

- What is the definition of the cross product of two vectors? Is it a scalar or vector?
- What can you say about vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ (all possibilities) if the cross product of these two vectors is a zero vector?

1. Determinant of a Matrix

A 2 × 2 matrix and 3 × 3 matrix of real numbers are of the forms:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \text{respectively.}$$

The **determinant** of a 2 × 2 matrix of real numbers is defined by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21},$$

and the **determinant** of a 3 × 3 matrix of real numbers is defined by

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Note that the matrix $\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$ is obtained from $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ by deleting the 1st row and the 1st column, $\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$ is obtained from $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ by deleting the 1st row and the 2nd column, and $\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ is obtained from $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ by deleting the 1st row and the 3rd column.

Example *Evaluate the determinants* $\begin{vmatrix} 1 & -2 \\ -3 & 4 \end{vmatrix}$, *and* $\begin{vmatrix} -1 & 0 & 2 \\ 5 & -4 & 3 \\ -6 & 0 & -8 \end{vmatrix}$.

$$\begin{vmatrix} 1 & -2 \\ -3 & 4 \end{vmatrix} = (1)(4) - (-2)(-3) = -2, \quad \begin{vmatrix} 5 & 3 \\ -6 & -8 \end{vmatrix} = -40 + 18 = -22$$

$$\begin{vmatrix} -1 & 0 & 2 \\ 5 & -4 & 3 \\ -6 & 0 & -8 \end{vmatrix} = (-1)(-2) - 0 + 2(-22) = -42$$

2. **The Cross Product**

Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$, and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. Then **the cross product** of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are defined by

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{\mathbf{k}}.$$

Example *Let* $\vec{\mathbf{u}} = \langle 1, -2, 3 \rangle$ *and* $\vec{\mathbf{v}} = \langle 4, 1, 2 \rangle$. *Compute* $\vec{\mathbf{w}} = \vec{\mathbf{u}} \times \vec{\mathbf{v}}$ *and check* $\vec{\mathbf{w}} \cdot \vec{\mathbf{u}}$ *and* $\vec{\mathbf{w}} \cdot \vec{\mathbf{v}}$.

$$\vec{\mathbf{w}} = \vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & -2 & 3 \\ 4 & 1 & 2 \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ 1 & 2 \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} \vec{\mathbf{k}}$$

$$= (-4-3)\vec{\mathbf{i}} - (2-12)\vec{\mathbf{j}} + (1+8)\vec{\mathbf{k}} = -7\vec{\mathbf{i}} + 10\vec{\mathbf{j}} + 9\vec{\mathbf{k}}$$

$$\vec{\mathbf{w}} \cdot \vec{\mathbf{u}} = \left(-7\vec{\mathbf{i}} + 10\vec{\mathbf{j}} + 9\vec{\mathbf{k}}\right) \cdot \left(\vec{\mathbf{i}} - 2\vec{\mathbf{j}} + 3\vec{\mathbf{k}}\right) = (-7) + (-20) + 27 = 0$$

$$\vec{\mathbf{w}} \cdot \vec{\mathbf{v}} = \left(-7\vec{\mathbf{i}} + 10\vec{\mathbf{j}} + 9\vec{\mathbf{k}}\right) \cdot \left(4\vec{\mathbf{i}} + \vec{\mathbf{j}} + 2\vec{\mathbf{k}}\right) = (-28) + 10 + 18 = 0$$

Note that $\vec{\mathbf{w}}$ is orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ and therefore it is orthogonal to the plane formed by vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

3. **Properties of the Cross Product**

Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ be vectors in V_3 and c be a scalar. Then

- **a.** $\vec{\mathbf{u}} \times \vec{\mathbf{u}} = \vec{\mathbf{0}}$
- **b.** $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}})$
- **c.** $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \vec{\mathbf{w}}$
- **d.** $(\vec{\mathbf{u}} + \vec{\mathbf{v}}) \times \vec{\mathbf{w}} = \vec{\mathbf{u}} \times \vec{\mathbf{w}} + \vec{\mathbf{v}} \times \vec{\mathbf{w}}$
- **e.** $(c\vec{\mathbf{u}}) \times \vec{\mathbf{v}} = c(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = \vec{\mathbf{u}} \times (c\vec{\mathbf{v}})$
- **f.** $\vec{\mathbf{0}} \times \vec{\mathbf{u}} = \vec{\mathbf{u}} \times \vec{\mathbf{0}} = \vec{\mathbf{0}}$
- **g.** $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{w}}$
- **h.** $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \cdot \vec{\mathbf{w}})\vec{\mathbf{v}} - (\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})\vec{\mathbf{w}}$

Show Property *g*. Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$, $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$, and $\vec{\mathbf{w}} = \langle w_1, w_2, w_3 \rangle$.

$$\begin{aligned}
\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{w}}) &= \langle u_1,u_2,u_3\rangle\cdot\begin{vmatrix}\vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}}\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}\\
&= \langle u_1,u_2,u_3\rangle\cdot\left(\begin{vmatrix} v_2 & v_3\\ w_2 & w_3\end{vmatrix}\vec{\mathbf{i}}-\begin{vmatrix} v_1 & v_3\\ w_1 & w_3\end{vmatrix}\vec{\mathbf{j}}+\begin{vmatrix} v_1 & v_2\\ w_1 & w_2\end{vmatrix}\vec{\mathbf{k}}\right)\\
&= u_1(v_2w_3-v_3w_2)-u_2(v_1w_3-v_3w_1)+u_3(v_1w_2-v_2w_1)\\
&= w_1(u_2v_3-v_2u_3)-w_2(u_1v_3-u_3v_1)+w_3(u_1v_2-u_2v_1)\\
&= \left(\begin{vmatrix} u_2 & u_3\\ v_2 & v_3\end{vmatrix}\vec{\mathbf{i}}-\begin{vmatrix} u_1 & u_3\\ v_1 & v_3\end{vmatrix}\vec{\mathbf{j}}+\begin{vmatrix} u_1 & u_2\\ v_1 & v_2\end{vmatrix}\vec{\mathbf{k}}\right)\cdot\langle w_1,w_2,w_3\rangle\\
&= (\vec{\mathbf{u}}\times\vec{\mathbf{v}})\cdot\vec{\mathbf{w}}
\end{aligned}$$

Remarks:

- For any vector $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ in V_3, $\vec{\mathbf{u}}\times\vec{\mathbf{v}}$ is orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.
- If $\vec{\mathbf{u}}\neq c\vec{\mathbf{v}}$, then $\vec{\mathbf{u}}\times\vec{\mathbf{v}}$ is orthogonal to the plane formed by $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ and it satisfies the right-hand rule: If you align the fingers of your right hand along the vector $\vec{\mathbf{u}}$ and bend your fingers around in the direction of rotation from $\vec{\mathbf{u}}$ toward $\vec{\mathbf{v}}$, your thumb will point in the direction of $\vec{\mathbf{u}}\times\vec{\mathbf{v}}$.
- Two nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ in V_3 are parallel if and only if $\vec{\mathbf{u}}\times\vec{\mathbf{v}}=\vec{\mathbf{0}}$.

Example *Compute* (1) $\vec{\mathbf{i}}\times\vec{\mathbf{j}}$ (2) $\vec{\mathbf{i}}\times\vec{\mathbf{k}}$ (3) $\left(\vec{\mathbf{i}}\times\vec{\mathbf{k}}\right)\times\vec{\mathbf{k}}$ (4) $\vec{\mathbf{i}}\cdot\left(\vec{\mathbf{k}}\times\vec{\mathbf{j}}\right)$

Since $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, $\vec{\mathbf{k}}$ are orthogonal, we can use the right-hand rule to determine these cross or dot products.

(1) $\vec{\mathbf{i}}\times\vec{\mathbf{j}}=\vec{\mathbf{k}}$

(2) $\vec{\mathbf{i}}\times\vec{\mathbf{k}}=-\vec{\mathbf{j}}$

(3) $\left(\vec{\mathbf{i}}\times\vec{\mathbf{k}}\right)\times\vec{\mathbf{k}}=\left(-\vec{\mathbf{j}}\right)\times\vec{\mathbf{k}}=-\left(\vec{\mathbf{j}}\times\vec{\mathbf{k}}\right)=-\vec{\mathbf{i}}$

(4) $\vec{\mathbf{i}}\cdot\left(\vec{\mathbf{k}}\times\vec{\mathbf{j}}\right)=\vec{\mathbf{i}}\cdot\left(-\vec{\mathbf{i}}\right)=-\left|\left|\vec{\mathbf{i}}\right|\right|=-1$

Example *Find two unit vectors orthogonal to the vectors* $\vec{\mathbf{u}}=3\vec{\mathbf{i}}-\vec{\mathbf{j}}+\vec{\mathbf{k}}$, *and* $\vec{\mathbf{v}}=4\vec{\mathbf{j}}+\vec{\mathbf{k}}$.

We know that $\vec{\mathbf{u}}\times\vec{\mathbf{v}}$ is orthogonal to $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ and $\pm\dfrac{1}{||\vec{\mathbf{u}}\times\vec{\mathbf{v}}||}\,\vec{\mathbf{u}}\times\vec{\mathbf{v}}$ are two unit vectors.

$$\vec{\mathbf{u}}\times\vec{\mathbf{v}}=\begin{vmatrix}\vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}}\\ 3 & -1 & 1\\ 0 & 4 & 1\end{vmatrix}=\begin{vmatrix}-1 & 1\\ 4 & 1\end{vmatrix}\vec{\mathbf{i}}-\begin{vmatrix}3 & 1\\ 0 & 1\end{vmatrix}\vec{\mathbf{j}}+\begin{vmatrix}3 & -1\\ 0 & 4\end{vmatrix}\vec{\mathbf{k}}=-5\vec{\mathbf{i}}-3\vec{\mathbf{j}}+12\vec{\mathbf{k}}$$

$$||\vec{\mathbf{u}}\times\vec{\mathbf{v}}||=\sqrt{(-5)^2+(-3)^2+12^2}=\sqrt{178}$$

$$\vec{\mathbf{w}}=\frac{1}{\sqrt{178}}\left(-5\vec{\mathbf{i}}-3\vec{\mathbf{j}}+12\vec{\mathbf{k}}\right),\qquad \vec{\mathbf{z}}=-\frac{1}{\sqrt{178}}\left(-5\vec{\mathbf{i}}-3\vec{\mathbf{j}}+12\vec{\mathbf{k}}\right)$$

4. The Magnitude of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$

Here is a relation of the magnitude $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|$ of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ and the magnitudes $\|\vec{\mathbf{u}}\|$ and $\|\vec{\mathbf{v}}\|$ of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

Theorem *Let θ be the angle between nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Then*

$$\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| = \|\vec{\mathbf{u}}\|\ \|\vec{\mathbf{v}}\|\ \sin\theta.$$

Proof

$$\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|^2 = (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \overset{\text{property } g.}{=} \Big[\ (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \times \vec{\mathbf{u}}\ \Big] \cdot \vec{\mathbf{v}} \overset{\text{property } b.}{=} -\Big[\ \vec{u} \times (\vec{u} \times \vec{v})\ \Big] \cdot \vec{v}$$

$$\overset{\text{property } h.}{=} -\Big[\ (\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})\vec{\mathbf{u}} - (\vec{\mathbf{u}} \cdot \vec{\mathbf{u}})\vec{\mathbf{v}}\ \Big] \cdot \vec{\mathbf{v}}$$

$$= -((\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) - (\vec{\mathbf{u}} \cdot \vec{\mathbf{u}})(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}})) = \|\vec{\mathbf{u}}\|^2\ \|\vec{\mathbf{v}}\|^2 - (\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})^2$$

$$= \|\vec{\mathbf{u}}\|^2\ \|\vec{\mathbf{v}}\|^2 - \|\vec{\mathbf{u}}\|^2\|\vec{\mathbf{v}}\|^2\cos^2\theta = \|\vec{\mathbf{u}}\|^2\|\vec{\mathbf{v}}\|^2(1 - \cos^2\theta)$$

$$= \|\vec{\mathbf{u}}\|^2\|\vec{\mathbf{v}}\|^2\sin^2\theta$$

Since $0 \le \theta \le \pi$, $\sin\theta > 0$ and $\sqrt{\sin^2\theta} = \sin\theta$. So,

$$\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| = \|\vec{\mathbf{u}}\|\ \|\vec{\mathbf{v}}\|\sin\theta.$$

Remarks:

- We can use this formula to compute $\sin\theta$ or θ :

$$\sin\theta = \frac{\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|}{\|\vec{\mathbf{u}}\|\ \|\vec{\mathbf{v}}\|}$$

  However, we need to identify if θ is in **the first or the second quadrant** since the right-side of the formula is positive.

- The **area of a parallelogram** that is formed by the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ when they are not parallel is

$$A = 2\left(\frac{1}{2}bh\right) = \|\vec{\mathbf{u}}\|\ \|\vec{\mathbf{v}}\|\ \sin\theta = \|\vec{\mathbf{u}}\|\ \|\vec{\mathbf{v}}\|\left(\frac{\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|}{\|\vec{\mathbf{u}}\|\ \|\vec{\mathbf{v}}\|}\right) = \|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|$$

  Hence, we can compute the area of a parallelogram without finding first the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

We can also use this formula to find a formula for computing **the distance from a given point to a given line**. Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be nonzero vectors and θ be the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Then the **distance** d from the end point of $\vec{\mathbf{u}}$ to the vector $\vec{\mathbf{v}}$ is:

$$d = \|\vec{\mathbf{u}}\|\sin\theta = \frac{\|\vec{\mathbf{u}}\|\ \|\vec{\mathbf{v}}\|\ \sin\theta}{\|\vec{\mathbf{v}}\|} = \frac{\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|}{\|\vec{\mathbf{v}}\|}.$$

Now consider **a parallelepiped** and let its three adjacent edges be $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$. Then the **volume** of this solid is the **area** of the parallelogram formed by $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ **times** the **altitude** which is

$$\Big|\ \|\vec{\mathbf{w}}\|\ \sin\theta\ \Big| = \Big|\ \mathrm{comp}_{\vec{\mathbf{u}}\times\vec{\mathbf{v}}}\ \vec{\mathbf{w}}\Big|$$

$$\text{Volume } = \|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|\ \Big|\ \mathrm{comp}_{\vec{\mathbf{u}}\times\vec{\mathbf{v}}}\ \vec{\mathbf{w}}\Big| = \|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|\ \frac{|\vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}})|}{\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|} = |\vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}})|$$

Example *Let* $\vec{\mathbf{u}} = \langle 1, -2, 3\rangle$, $\vec{\mathbf{v}} = \langle -4, 1, 2\rangle$ *and* $\vec{\mathbf{w}} = \langle 2, -1, 2\rangle$.

(1) *Determine the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.*

(2) *Find the area of the parallelogram with two adjacent sides formed by the vectors* $\vec{\mathbf{u}} = \langle 1,-2,3\rangle$, *and* $\vec{\mathbf{v}} = \langle -4,1,2\rangle$.

(3) *Find the volume of the parallelepiped with three adjacent sides formed by* $\vec{u}$, $\vec{v}$ *and* $\vec{w}$.

$$\vec{\mathbf{u}}\times\vec{\mathbf{v}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & -2 & 3 \\ -4 & 1 & 2 \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ 1 & 2 \end{vmatrix}\vec{\mathbf{i}} - \begin{vmatrix} 1 & 3 \\ -4 & 2 \end{vmatrix}\vec{\mathbf{j}} + \begin{vmatrix} 1 & -2 \\ -4 & 1 \end{vmatrix}\vec{\mathbf{k}} = -7\vec{\mathbf{i}} - 14\vec{\mathbf{j}} - 7\vec{\mathbf{k}}$$

$||\vec{\mathbf{u}}\times\vec{\mathbf{v}}|| = \sqrt{(-7)^2+(-14)^2+(-7)^2} = 7\sqrt{6} = 17.14643$

$||\vec{\mathbf{u}}|| = \sqrt{1^2+(-2)^2+3^2} = \sqrt{14}$, $\quad ||\vec{\mathbf{v}}|| = \sqrt{(-4)^2+1^2+2^2} = \sqrt{21}$

(1)

$$\sin\theta = \frac{||\vec{\mathbf{u}}\times\vec{\mathbf{v}}||}{||\vec{\mathbf{u}}||\ \ ||\vec{\mathbf{v}}||} = \frac{7\sqrt{6}}{\sqrt{14}\sqrt{21}} = 1,\quad \theta = \frac{\pi}{2}\quad (\vec{u}\cdot\vec{v} = 0)$$

(2)

$$\text{Area of the parallelogram} = ||\vec{\mathbf{u}}\times\vec{\mathbf{v}}|| = 17.14643.$$

(3)

$$\vec{\mathbf{w}}\cdot(\vec{\mathbf{u}}\times\vec{\mathbf{v}}) = \langle 2,-1,2\rangle\cdot\langle -7,-14,-7\rangle = -14+14-14 = -14$$

$$\text{Volume of the parallelepiped } = |-14| = 14$$

Example *Find the distance from the point* $Q(1,2,1)$ *to the line through the points* $P(2,1,-3)$ *and* $R(2,-1,3)$.

$\overrightarrow{PQ} = \langle -1,1,4\rangle$, $\overrightarrow{PR} = \langle 0,-2,6\rangle$,

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ -1 & 1 & 4 \\ 0 & -2 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ -2 & 6 \end{vmatrix}\vec{\mathbf{i}} - \begin{vmatrix} -1 & 4 \\ 0 & 6 \end{vmatrix}\vec{\mathbf{j}} + \begin{vmatrix} -1 & 1 \\ 0 & -2 \end{vmatrix}\vec{\mathbf{k}} = 14\vec{\mathbf{i}} + 6\vec{\mathbf{j}} + 2\vec{\mathbf{k}}$$

$$d = \frac{\left|\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|\right|}{\left|\left|\overrightarrow{PR}\right|\right|} = \frac{\sqrt{14^2+6^2+2^2}}{\sqrt{0+(-2)^2+6^2}} = 2.428992$$

Example *If we apply a force of magnitude 20 pounds at the end of a 15-inch-long wrench, at an angle of* $\frac{\pi}{6}$ *to the wrench, find the magnitude of the torque applied to the bolt. What is the maximum torque that a force of 20 pounds applied at that point can produce?*

15 inches $=\frac{15}{12}$ feet

$\left\|\vec{\tau}\right\| = \left\|\vec{F}\right\|\,\|\vec{r}\|\sin\theta = 20(\frac{15}{12})\sin\left(\frac{\pi}{6}\right) = \frac{25}{2} = 12.5$ foot-pounds

When $\theta = \frac{\pi}{2}$, $\sin\theta = 1$ and $\left\|\vec{\tau}\right\| = 20(\frac{15}{12}) = 25$ foot-pounds