Applications of First Order Differential Equations - (1.3)(2.7)(2.8)

1. Growth and Decay:
Consider the initial value problem:
\[ \frac{dP}{dt} = kP, \quad P(0) = P_0. \]
Function \( P(t) \) represents population at the time \( t \). When \( k > 0 \), the population is increasing and when \( k < 0 \), the population is decreasing. The equation is separable and solution is
\[ P(t) = P_0 e^{kt}. \]

Logistic Equation:
\[ \frac{dP}{dt} = P(a - bP), \quad P(0) = P_0. \]
Logistic curves have proved to be quite accurate in predicting the growth patterns, in a limited space, of certain types of bacteria, water fleas, and fruit flies. The equation is separable, and also a Bernoulli equation. The solution is:
\[ P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}. \]

**Example** Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number \( x \) of infected students but also to the number of students not infected, determine the number of infected students after 6 days if it is further observed that after 4 days 50 students are infected.

The mathematical model is:
\[ \frac{dx}{dt} = kx(1000 - x), \quad x(0) = 1. \]
Since
\[ \frac{dx}{dt} = 1000kx - kx^2, \quad a = 1000k, \quad b = k, \quad P_0 = 1. \]
Hence
\[ x(t) = \frac{1000k}{k + (1000k - k)e^{-1000kt}} = \frac{1000}{1 + 999e^{-1000kt}}. \]
It is known that \( x(4) = 50 \). Then solve \( k \) from the equation:
\[ 50 = \frac{1000}{1 + 999e^{-1000k(4)}} \Rightarrow k = -\frac{1}{4000} \ln \frac{19}{999} \]
\[ x(t) = \frac{1000}{1 + 999e^{0.25 \ln(19/999) t}}, \quad x(6) = \frac{1000}{1 + 999e^{0.25 \ln(19/999)(6)}} = 276 \]
About 276 students are infected after 6 days.

2. Newton’s Law of Cooling:
\[ \frac{dT}{dt} = k(T - T_m), \quad T(0) = T_0. \]
Function \( T(t) \) represents temperature of an object at the time \( t \) where \( T_0 \) is the initial temperature of the object and \( T_m \) is the temperature surrounded the object. When \( k > 0 \), \( T \) is increasing and when \( k < 0 \), \( T \) is decreasing. The equation is separable and the solution is:
\[ T(t) = T_m + e^{kt}(T_0 - T_m). \]

**Example** A small metal bar, whose initial temperature was 20°C, is dropped into a container of boiling water. How long will it take the bar to reach 90°C if is known that its temperature increased 2°C in 1 second? How long will it take the bar to reach 98°C?

Let \( T \) be the temperature of the bar at the \( t \). Then we know: \( T_m = 100°C, \quad T_0 = 20°C \). So,
\[ T(t) = 100 + (20 - 100)e^{kt} = 100 - 80e^{kt}. \]

Since \( T(1) = 2 + 20 = 22 \text{°C}, \) solve \( k \) from the equation:

\[ 22 = 100 - 80e^k. \]

\[ k = \ln \frac{39}{40} \quad \text{and} \quad T(t) = 100 - 80e^{\ln(\frac{39}{40})t}. \]

Find \( t \) when \( T(t) = 90 \text{°C}. \)

\[ 90 = 100 - 80e^{\ln(\frac{39}{40})t} \Rightarrow t = -\frac{\ln \frac{8}{80}}{\ln \frac{39}{40}} = 82.13 \text{ seconds} \]

Find \( t \) when \( T(t) = 98 \text{°C}. \)

\[ 98 = 100 - 80e^{\ln(\frac{39}{40})t} \Rightarrow t = -\frac{\ln \frac{40}{80}}{\ln \frac{39}{40}} = 145.70 \text{ seconds} \]

3. LR, and RC Circuits:

\[ L \frac{dI}{dt} + RI = E(t), \quad I(0) = I_0 \]

\[ R \frac{dQ}{dt} + \frac{1}{C}Q = E(t), \quad Q(0) = Q_0 \]

Functions \( I(t) \) and \( Q(t) \) are current and charge at the time \( t \), respectively. \( L, \ R, \ \) and \( C \) are constants for inductor, resistor and capacitor, respectively. Function \( E(t) \) is voltage on the circuit at the time \( t \). By Kirchhoff’s second law,

\[ E_R = RI, \quad E_L = L \frac{dI}{dt}, \quad E_C = \frac{1}{C} \int I(t)dt, \quad \frac{dQ}{dt} = Q \]

These two equations are linear (in \( I \) and \( Q \)). The solutions depend on given \( E(t) \).

**Example** Consider a LR-circuit with \( L = 4 \), \( R = 2 \), \( I(0) = 0 \), and \( E(t) = \begin{cases} \ t & \text{if } 0 \leq t \leq 1 \\ 1 & \text{if } t > 1 \end{cases} \). Solve \( I(t) \).

Solve \( I(t) \) from the initial value problem:

\[ 4 \frac{dI}{dt} + 2I = E(t), \quad I(0) = 0 \Rightarrow \frac{dI}{dt} + \frac{1}{2}I = \frac{1}{4}E(t), \quad I(0) = 0 \]

It is linear in \( I \).

a. \( h(t) = \int \frac{1}{4} dt = \frac{1}{4} t \), I.F.: \( e^{\frac{t^2}{2}} \)

b. \( k(t) = \int e^{\frac{t^2}{2}} dt = 2e^{\frac{t^2}{2}}t - 4e^{\frac{t^2}{2}}, \ 0 \leq t \leq 1 \)

c. \( I(t) = e^{-\frac{t^2}{2}}(C_1 + 2te^{\frac{t^2}{2}} - 4e^{\frac{t^2}{2}}), \ 0 \leq t \leq 1 \)

d. When \( t = 0 \), \( I(0) = 0 \). Solve \( C_1 : \)

\[ 0 = e^0(C_1 + 2e^0(0) - 4e^0), \quad C_1 = 4. \]

\[ I(t) = e^{-\frac{t^2}{2}}(4 + 2e^{\frac{t^2}{2}} - 4e^{\frac{t^2}{2}}), \ 0 \leq t \leq 1 \]

In particular, \( I(1) = e^{-\frac{1}{2}}(4 + 2e^{\frac{1}{2}} - 4e^{\frac{1}{2}}) = 4e^{-\frac{1}{2}} + 2 - 4 = 4e^{-\frac{1}{2}} - 2. \)

When \( t = 1 \), \( I(1) = 4e^{-\frac{1}{2}} - 2. \) Solve \( C_2 : \)

\[ 4e^{-\frac{1}{2}} - 2 = e^{-\frac{1}{2}}(C_2 + 2e^{\frac{1}{2}}) \Rightarrow 4 - 2e^{\frac{1}{2}} = C_2 + 2e^{\frac{1}{2}}, \quad C_2 = 4 - 4e^{\frac{1}{2}} \]

\[ I(t) = e^{-\frac{t^2}{2}}(4 - 4e^{\frac{1}{2}} + 2e^{\frac{t^2}{2}}), \ t > 1 \]

Solution:
\[ I(t) = \begin{cases} 
 e^{-\frac{t}{2}}(4 + 2te^{\frac{t}{2}} - 4e^{\frac{t}{2}}), & 0 \leq t \leq 1 \\
 e^{-\frac{t}{2}}(4 - 4e^{\frac{1}{2}} + 2e^{\frac{1}{2}}), & t > 1 
\end{cases} \]