1. **Separable Equations:**
   A first order differential equation is said to be **separable** if it is of the form
   \[
   \frac{dy}{dx} = g(x)h(y)
   \]

2. **Method of Separation of Variables:**
   Observe that a **separable equation** can be written as
   \[
   \int \frac{1}{h(y)} dy = \int g(x)dx
   \]
   If we know the antiderivatives of \( \frac{1}{h(y)} \) and \( g(x) \) are \( H(y) \) and \( G(x) \), then
   \[
   H(y) = G(x) + C
   \]
   is the general solution of the differentiation equation.

**Example** Solve \((1 + x)dy - ydx = 0\)

\[
\frac{1}{y} dy = \frac{1}{1 + x} dx
\]

\[
\int \frac{1}{y} dy = \int \frac{1}{1 + x} dx, \quad \ln|y| = \ln|1 + x| + C
\]

The general solution (in an explicit form):
\[
y = Ce^{\ln|1+x|} = C(1 + x)
\]

**Example** Solve the initial value problem:
\[
\cos(x)(e^{2y} - y) \frac{dy}{dx} = e^y \sin(2x), \quad y(0) = 0
\]

\[
\frac{e^{2y} - y}{e^y} dy = \frac{\sin(2x)}{\cos(x)} dx = \frac{2\sin x \cos x}{\cos x} dx = 2 \sin x \ dx
\]

\[
\int (e^y - ye^{-y})dy = 2 \int \sin(x)dx
\]

The general solution (in an implicit form):
\[
e^y + ye^{-y} + e^{-y} = -2 \cos(x) + C
\]

Now to find the solution for the initial-value problem: let \( x = 0 \), and \( y = 0 \)

\[
1 - 0 + 1 = -2 \cos(0) + C, \quad C = 4
\]

the solution for initial value problem:
\[
e^y + ye^{-y} + e^{-y} = -2 \cos(x) + 4
\]

**Example** Solve \( \frac{dy}{dx} = y^2 - 4. \)

\[
\frac{1}{y^2 - 4} dy = dx \quad \text{if} \quad y^2 - 4 \neq 0
\]
\[
\int \frac{dy}{(y+2)(y-2)} = \int dx = x + C \\
\int \frac{dy}{(y+2)(y-2)} = \frac{1}{4} \int \left( \frac{1}{y-2} - \frac{1}{y+2} \right) dx = \frac{1}{4} \ln|y-2| - \ln|y+2| = \frac{1}{4} \ln\left( \frac{y-2}{y+2} \right)
\]

\[
\frac{1}{4} \ln\left( \frac{y-2}{y+2} \right) = x + C, \quad \ln\left( \frac{y-2}{y+2} \right) = 4(x + C), \quad \frac{y-2}{y+2} = e^{4(x+C)} = Ce^{4x}
\]

\[
\frac{y-2}{y+2} = \frac{y+2-4}{y+2} = 1 - \frac{4}{y+2} = Ce^{4x}
\]

\[
\frac{4}{y+2} = 1 - Ce^{4x}, \quad y + 2 = \frac{4}{1 - Ce^{4x}}
\]

The general solution (in an explicit form):

\[
y = \frac{4}{1 - Ce^{4x}} - 2 = \frac{4 - 2 + 2Ce^{4x}}{1 - Ce^{4x}} = 2 \frac{1 + Ce^{4x}}{1 - Ce^{4x}}
\]

If \(y^2 - 4 = 0\), then \(y = \pm 2\). Observe that in this case, \(\frac{dy}{dx} = 0\) or \(y = C\).

**Example** \(\frac{dy}{dx} = \frac{xy + 2y - x - 2}{xy - 3x + x - 3}\)

\[
\frac{dy}{dx} = \frac{y(x+2) - (x+2)}{y(x-3) + (x-3)} = \frac{(y-1)(x+2)}{(y+1)(x-3)}, \quad \frac{y+1}{y-1} dy = \frac{x+2}{x-3} dx
\]

\[
\int \frac{y+1}{y-1} dy = \int \frac{x+2}{x-3} dx = \int \left(1 + \frac{5}{x-3}\right) dx
\]

The general solution (in implicit form):

\[
y + 2 \ln(y-1) = x + 5 \ln(x-3) + C
\]

**Example** \(e^y \frac{dy}{dx} = e^{-y} + e^{-2x-y}\)

\[
e^y \frac{dy}{dx} = e^{-y}(1 + e^{-2x}), \quad \frac{y}{e^y} dy = \frac{(1 + e^{-2x})}{e^x} dx, \quad ye^y dy = (e^x + e^{-x}) dx
\]

\[
\int ye^y dy = \int (e^x + e^{-x}) dx, \quad \int ye^y dy = ye^y - e^y + C, \quad \int (e^x + e^{-x}) dx = e^x - e^{-x} + C
\]

The general solution (in implicit form):

\[
ye^y - e^y = e^x - e^{-x} + C
\]