5.1.2 Power Series Solutions

Consider a **linear second order differential equation** of the form:
\[ a_2(x)y'' + a_1(x)y' + a_0(x)y = 0. \]  
(1)

Rewrite it in standard form:
\[ y'' + P(x)y' + Q(x)y = 0 \quad \left( P(x) = \frac{a_1(x)}{a_2(x)}, \quad Q(x) = \frac{a_0(x)}{a_2(x)} \right) \]  
(2)

Recall (Page 104-109) that the general solution of the equation (1) is
\[ y = c_1 y_1 + c_2 y_2 \]
where \( c_1 \) and \( c_2 \) are arbitrary constants, and \( y_1 \) and \( y_2 \) are solutions of (1) and are linearly independent.

1. **Ordinary Points and Singular Points**:
   
   **Analytic functions**: A real-valued function \( f(x) \) is **analytic** at \( x_0 \) if it has a **power series representation** at \( x_0 \), i.e., \( f \) can be written as
   \[ f(x) = \sum_{m=0}^{\infty} c_m (x - x_0)^m \quad \text{for} \quad x \text{ in a neighborhood of} \ x_0. \]

   Observe that if \( f^{(n)}(x) \) exists for any \( n \) for \( x \) in the neighborhood of \( x_0 \) then \( f(x) \) has a power series in the powers of \( x - x_0 \) that can be derived by the Taylor series of \( f(x) \) at \( x = x_0 \).

   **Ordinary Points and Singular Points**:
   
   a. **Ordinary Points**: A point \( x_0 \) is said to be an **ordinary point** of the differential equation (1) if both \( P(x) \) and \( Q(x) \) in (2) are **analytic** at \( x_0 \).

   b. **Singular Points**: A point that is **not an ordinary point** is said to be a **singular point** of the equation. Observe that when \( a_0(x) \), \( a_1(x) \) and \( a_2(x) \) are **polynomials** \( P(x) \) and \( Q(x) \) are analytic everywhere except where \( a_2(x) = 0 \).

   **Example** Find all singular points of the differential equation.
   \[ x(x^2 - 1)y'' + (x^2 + 1)y' - \cot(x) \ y = 0 \]

   \[ a_2(x) = x(x^2 - 1), \quad a_1(x) = x^2 + 1, \quad a_0(x) = -\cot(x) = -\frac{\cos(x)}{\sin(x)} \]

   \[ P(x) = \frac{x^2 + 1}{x(x^2 - 1)} = \frac{-x}{x^2 - 1}, \quad Q(x) = \frac{-x}{x(x^2 - 1)} = -\frac{1}{x^2 - 1} \]

   \[ a_2(x) = 0 \text{ at } x = 0, -1, \text{ and } 1. \ a_0(x) = -\cot(x) = -\frac{\cos(x)}{\sin(x)} \text{ is not defined when } \sin(x) = 0, \text{ i.e.} \]

   \[ x = \pm n\pi \text{ where } n = 0, 1, 2, \ldots. \]

   Hence, the singular points of the equation are \( x = 0, -1, 1 \) and \( \pm n\pi \text{ where } n = 0, 1, 2, \ldots. \)

2. **Existence of Power Series Solutions**:
   Let \( x_0 \) be an **ordinary point** of the differential equation \( a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \). Then we can always **find two linearly independent solutions in the form of a power series in the powers of** \( x - x_0 \). There exists \( R > 0 \) such that the series solution \( y = \sum_{m=0}^{\infty} c_m (x - x_0)^m \) converges for \( |x - x_0| < R \). Note that \( R \) is the **distance** from \( x_0 \) to the **closest singular point** of the equation.

   **Example** Determine if a power series solution of the equation \( (x^2 - 3x - 4)y'' + xy' - y = 0 \) exists. If so,
find the radius $R$ of convergence of the power series.

$$a_2(x) = x^2 - 3x - 4 = (x - 4)(x + 1) = 0, \ x = -1, 4.$$ Let $x_0 = 0.$ $x_0$ is an ordinary point of the equation. So, the equation has a power series solution $y = \sum_{m=0}^{\infty} c_m x^m$. The closest distance from 0 to $-1$ and 4 is 1 so $R = 1$.

3. Finding Power Series Solutions:

For a given differential equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, how can we find its power series solution assuming it exists?

Steps:

a. Choose an ordinary point $x_0$ of the equation and let $y = \sum_{m=0}^{\infty} c_m (x-x_0)^m$. Compute

$$y' = \sum_{m=1}^{\infty} c_m m (x-x_0)^{m-1}, \quad y'' = \sum_{m=2}^{\infty} c_m m (m-1) (x-x_0)^{m-2} .$$

b. Substitute $y$, $y'$ and $y''$ into the differential equation and simplify it to a single sum.

c. Solve the recurrence relation for $c_m$ (relate $c_m$ to $c_k$ where $k < m$) and express $c_m$ in terms of $m$ if possible (write $c_m$ as a function of $m$).

d. Write the solution $y$ in either a closed form (the limit of the power series), a power series with a general term, or a power series with a few nonzero terms.

Example Find a power series solution of the equation $y'' + xy = 0$.

Since $a_2(x) = 1 \neq 0$ everywhere and $a_1(x) = 0$ and $a_0(x) = x$ are polynomials, the differential equation does not have any singular point.

a. Let $x_0 = 0$, and

$$y = \sum_{m=0}^{\infty} c_m x^m .$$

Then

$$y' = \sum_{m=1}^{\infty} c_m m x^{m-1}, \quad y'' = \sum_{m=2}^{\infty} c_m m (m-1) x^{m-2} .$$

b. Substitute $y$, $y'$ and $y''$ into the differential equation and simplify it to a single sum.

$$y'' + xy = \sum_{m=2}^{\infty} c_m m (m-1) x^{m-2} + x \sum_{m=0}^{\infty} c_m x^m$$

$$= \sum_{m=2}^{\infty} c_m m (m-1) x^{m-2} + \sum_{m=0}^{\infty} c_m x^{m+1} = 0$$

Replace $m$ by $k$ in the 1st summation and replace $m + 1$ by $k - 2$ (or $m = k - 3$, or $k = m + 3$) in the 2nd summation (when $m = 0, k = 3$):

$$\sum_{k=2}^{\infty} c_k (k-1) x^{k-2} + \sum_{k=3}^{\infty} c_{k-3} x^{k-2} = c_2 (2)(2-1) + \sum_{k=3}^{\infty} \left[ c_k (k-1) + c_{k-3} \right] x^{k-2}$$

$$= 2c_2 + \sum_{k=3}^{\infty} \left[ c_k (k-1) + c_{k-3} \right] x^{k-2} = 0$$

c. $c_2 = 0$, and $c_k (k-1) + c_{k-3} = 0$ for $k = 3, 4, 5, \ldots \Rightarrow$

$$c_k = -\frac{1}{k(k-1)} c_{k-3}, \ k = 3, 4, 5, \ldots$$

Since $c_2 = 0$ and $c_5, c_8, \ldots$ depends on $c_2$, $c_{3m+2} = 0$ for $m = 0, 1, \ldots$ $c_0$ and $c_1$ are free, and $c_3, c_6, \ldots, c_{3m}$ depend on $c_0$ and $c_4, c_7, \ldots, c_{3m+1}$ depend on $c_1.$
\[
\begin{array}{|c|c|}
\hline
k = 3 & c_3 = -\frac{1}{(3)(2)}c_0 = -\frac{1}{6}c_0 \\
\hline
k = 6 & c_6 = -\frac{1}{(6)(5)}c_3 = \frac{(-1)^2}{(6)(5)(3)(2)}c_0 = \frac{1}{180}c_0 \\
\hline
k = 9 & c_9 = -\frac{1}{(9)(8)}c_6 = \frac{(-1)^3}{(9)(8)(6)(5)(3)(2)}c_0 = -\frac{1}{12960}c_0 \\
\hline
\vdots & \vdots \\
\hline
k = 3m & c_{3m} = \frac{(-1)^m}{(3m)(3m-1)\ldots(9)(8)(6)(5)(3)(2)}c_0 \\
\hline
\end{array}
\]

d. 
\[
y = \sum_{m=0}^{\infty} c_m x^m = \sum_{k=0}^{\infty} c_{3k} x^{3k} + \sum_{k=0}^{\infty} c_{3k+1} x^{3k+1} + \sum_{k=0}^{\infty} c_{3k+2} x^{3k+2} = \sum_{k=0}^{\infty} c_{3k} x^{3k} + \sum_{k=0}^{\infty} c_{3k+1} x^{3k+1} \\
= c_0 \left(1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \frac{1}{12960}x^9 + \ldots\right) + c_1 \left(x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \frac{1}{45360}x^{10} + \ldots\right) \\
= c_0 \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^m}{(3m)(3m-1)\ldots(9)(8)(6)(5)(3)(2)} x^{3k}\right) \\
+ c_1 \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^m}{(3m+1)(3m+1)\ldots(10)(9)(7)(6)(4)(3)} x^{3k+1}\right) \\
\]

Example  Solve \((x^2 + 1)y'' + xy' - y = 0, y(0) = 1, y'(0) = -2\) using the Power Series Method.

\(a_2(x) = x^2 + 1 \neq 0\) for all \(x\) and \(a_1(x) = x\) and \(a_0(x) = -1\) are polynomials. So, the differential equation has no singular point.

a. Let \(x_0 = 0\), and and \(y = \sum_{m=0}^{\infty} c_m x^m\). Then

\[
y' = \sum_{m=1}^{\infty} c_m m x^{m-1}, \quad y'' = \sum_{m=2}^{\infty} c_m m(m-1) x^{m-2}.
\]

b. \((x^2 + 1)y'' + xy' - y = x^2y'' + y'' + xy' - y\)

\[
x^2 \sum_{m=2}^{\infty} c_m m(m-1) x^{m-2} + \sum_{m=2}^{\infty} c_m m(m-1) x^{m-2} + x \sum_{m=1}^{\infty} c_m m x^{m-1} - \sum_{m=0}^{\infty} c_m x^m \\
= \sum_{m=2}^{\infty} c_m m(m-1) x^{m} + \sum_{m=2}^{\infty} c_m m(m-1) x^{m-2} + \sum_{m=1}^{\infty} c_m m x^{m} - \sum_{m=0}^{\infty} c_m x^m
\]

Replace \(m - 2\) by \(k\) or \(m\) by \(k + 2\) in the 2nd summation and replace \(m\) by \(k\) in the 1st, 3rd and 4th
summation:
\[
\sum_{k=2}^{\infty} c_k(k-1)k^k + \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} c_k k^k - \sum_{k=0}^{\infty} c_k x^k = 0
\]
\[
c_2(2) + c_3(3)2 + c_1 x - c_0 - c_1 x + \sum_{k=2}^{\infty} [c_k(k-1) + c_{k+2}k+2(k+1) + c_k - c_k] = 0
\]
\[
(2c_2 - c_0) + 6c_3 x + \sum_{k=2}^{\infty} [c_k(k^2 - 1) + c_{k+2}k(k+1) + c_k - c_k] = 0
\]

c. \(2c_2 - c_0 = 0, \ c_2 = \frac{1}{2}c_0\)

\(6c_3 = 0, \ c_3 = 0\)

\(c_0\) and \(c_1\) are free.

\(c_k(k^2 - 1) + c_{k+2}(k+2)(k+1) = 0\) for \(k = 2, 3, \ldots \Rightarrow\)

\[c_{k+2} = -\frac{(k-1)(k+1)}{(k+2)(k+1)} \frac{k-1}{k+2} c_k, \ k = 2, 3, \ldots\]

Since \(c_3 = 0\), and \(c_5, c_7, \ldots, c_{2m+1}, \ldots\) depend on \(c_3\), \(c_{2m+1} = 0\) for \(m = 1, 2, \ldots\)

<table>
<thead>
<tr>
<th>(k)</th>
<th>(c_k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(-\frac{1}{4}c_2 = -\frac{1}{4} \left(\frac{1}{2}c_0\right) = -\frac{1}{8}c_0)</td>
</tr>
<tr>
<td>4</td>
<td>(-\frac{3}{6}c_4 = -\frac{3}{6} \left(\frac{-1}{8}c_0\right) = \frac{1}{16}c_0)</td>
</tr>
<tr>
<td>6</td>
<td>(-\frac{5}{8}c_6 = -\frac{5}{8} \left(\frac{-1}{16}c_0\right) = -\frac{5}{128}c_0)</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>2m</td>
<td>(c_{2m} = (-1)^m \frac{(3)(5)\ldots(2m-3)}{2^{m-1}(m)!}, \ m = 2, 3, \ldots)</td>
</tr>
</tbody>
</table>

d. \(y = \sum_{m=0}^{\infty} c_m x^m = \sum_{m=0}^{\infty} c_{2m} x^{2m} + \sum_{m=0}^{\infty} c_{2m+1} x^{2m+1}\)

\[
y = c_0 \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{16}x^6 + \frac{5}{128}x^8 + \ldots\right) + c_1 x
\]

\[
y = c_0 \left(1 + \frac{1}{2}x^2 + \sum_{m=2}^{\infty} (-1)^m \frac{(3)(5)\ldots(2m-3)}{2^{m-1}(m)!} x^{2m} \right) + c_1 x
\]

e. \(y(0) = c_0 = 1, \ y'(0) = c_0 \left(0 + x - \frac{1}{2}x^3 + \frac{3}{8}x^5 + \ldots\right) + c_1, \ y'(0) = c_1 = -2\)

\(y = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \frac{5}{128}x^8 + \ldots - 2x\)
Example  Solve $y'' - (1 + x)y' + y = 0$

$a_2(x) = 1 \neq 0$ for all $x$ and $a_1(x) = -(1 + x)$ and $a_0(x) = 1$ are polynomials. So, the equation has no singular point.

a. Let $x_0 = 0$, and and $y = \sum_{m=0}^{\infty} c_m x^m$. Then

$$y' = \sum_{m=1}^{\infty} c_m m x^{m-1}, \quad y'' = \sum_{m=2}^{\infty} c_m m (m-1) x^{m-2}.$$ 

b. $y'' - (1 + x)y' + y = y'' - y' - xy' + y = 0$

$$\sum_{m=2}^{\infty} c_m m (m-1) x^{m-2} - \sum_{m=1}^{\infty} c_m m x^{m-1} - x \sum_{m=1}^{\infty} c_m m x^{m-1} + \sum_{m=0}^{\infty} c_m x^m$$

$$= \sum_{m=2}^{\infty} c_m m (m-1) x^{m-2} - \sum_{m=1}^{\infty} c_m m x^{m-1} - \sum_{m=1}^{\infty} c_m m x^m + \sum_{m=0}^{\infty} c_m x^m$$

Replace $m - 2$ by $k$ in the 1st summation, $m - 1$ by $k$ in the 2nd summation, and $m$ by $k$ in the 3rd and 4th summation.

$$\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_{k+1}(k+1)x^k - \sum_{k=1}^{\infty} c_k kx^k + \sum_{k=0}^{\infty} c_k x^k$$

$$= \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=0}^{\infty} c_{k+1}(k+1)x^k - \sum_{k=0}^{\infty} c_k kx^k + \sum_{k=0}^{\infty} c_k x^k$$

$$= c_2(2)(1) + \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+1)x^k - c_1(1) - \sum_{k=1}^{\infty} c_{k+1}(k+1)x^k - \sum_{k=1}^{\infty} c_k kx^k + c_0 + \sum_{k=0}^{\infty} c_k x^k$$

$$= 2c_2 - c_1 + c_0 + \sum_{k=1}^{\infty} [c_{k+2}(k+2)(k+1) - c_{k+1}(k+1) - c_k k + c_k] x^k$$

c. $2c_2 - c_1 + c_0 = 0$, $c_2 = \frac{c_0 - c_1}{2} = \frac{c_0}{2} - \frac{c_1}{2}$, $c_0$ and $c_1$ are free.

$$c_{k+2}(k+2)(k+1) - c_{k+1}(k+1) - c_k k + c_k = c_{k+2}(k+2)(k+1) - c_{k+1}(k+1) - c_k(k-1) = 0$$

for $k = 1, 2, \ldots$, implies

$$c_{k+2} = \frac{(k-1)c_k + (k+1)c_{k+1}}{(k+2)(k+1)} = \frac{k-1}{(k+2)(k+1)} c_k + \frac{1}{k+2} c_{k+1}, \quad k = 1, 2, 3, \ldots$$
\[ k = 1 \quad c_3 = \frac{1}{3}c_2 = \frac{1}{3} \left( \frac{c_0}{2} - \frac{c_1}{2} \right) = \frac{1}{6}c_0 - \frac{1}{6}c_1 \]

\[ k = 2 \quad c_4 = \frac{1}{(4)(3)}c_2 + \frac{1}{4}c_3 = \frac{1}{(4)(3)} \left( \frac{c_0}{2} - \frac{c_1}{2} \right) + \frac{1}{4} \left( \frac{1}{6}c_0 - \frac{1}{6}c_1 \right) = \frac{1}{12}c_0 - \frac{1}{12}c_1 \]

\[ k = 3 \quad c_5 = \frac{1}{(5)(4)}c_3 + \frac{1}{5}c_4 = \frac{1}{(5)(4)} \left( \frac{1}{6}c_0 - \frac{1}{6}c_1 \right) + \frac{1}{5} \left( \frac{1}{12}c_0 - \frac{1}{12}c_1 \right) = \frac{1}{40}c_0 - \frac{1}{40}c_1 \]

\[ k = 4 \quad c_6 = \frac{1}{(6)(5)}c_4 + \frac{1}{6}c_5 = \frac{1}{(6)(5)} \left( \frac{1}{12}c_0 - \frac{1}{12}c_1 \right) + \frac{1}{6} \left( \frac{1}{40}c_0 - \frac{1}{40}c_1 \right) = \frac{1}{144}c_0 - \frac{1}{144}c_1 \]

d.

\[
y = \sum_{m=0}^{\infty} c_m x^m = c_0 + c_1 x + \left( \frac{c_0}{2} - \frac{c_1}{2} \right) x^2 + \left( \frac{1}{6}c_0 - \frac{1}{6}c_1 \right) x^3 + \left( \frac{1}{12}c_0 - \frac{1}{12}c_1 \right) x^4 + \frac{1}{40}c_0 - \frac{1}{40}c_1 \right) x^5 + \left( \frac{1}{144}c_0 - \frac{1}{144}c_1 \right) x^6 + \ldots
\]

\[
= c_0 \left( 1 + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{12} x^4 + \frac{1}{40} x^5 + \frac{1}{144} x^6 + \ldots \right)
\]

\[
- c_1 \left( x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{12} x^4 + \frac{3}{40} x^5 + \frac{1}{144} x^6 + \ldots \right)
\]

**Example**  Solve the initial value problem \( y'' + x^2 y = 0 \), \( y(0) = 1 \), \( y'(0) = -1 \). Series solution is:

\( a_2(x) = 1 \neq 0 \) for all \( x \). So, the equation has no singular point.

a. Let \( x_0 = 0 \), and and \( y = \sum_{m=0}^{\infty} c_m x^m \). Then

\[
y' = \sum_{m=1}^{\infty} c_m m x^{m-1}, \quad y'' = \sum_{m=2}^{\infty} c_m m(m-1) x^{m-2}.
\]

b.

\[
y'' + x^2 y = \sum_{m=2}^{\infty} c_m m(m-1) x^{m-2} + x^2 \sum_{m=0}^{\infty} c_m x^m = \sum_{m=2}^{\infty} c_m m(m-1) x^{m-2} + \sum_{m=0}^{\infty} c_m x^{m+2} = 0
\]

Replace \( m \) by \( k \) in the 1st summation and \( m + 2 \) by \( k - 2 \) in the 2nd summation.

\[
\sum_{k=2}^{\infty} c_k k(k-1) x^{k-2} + \sum_{k=4}^{\infty} c_k x^{k-2} = 0
\]

\[
c_2(2)(1) + c_3(3)(2)x + \sum_{k=4}^{\infty} \left[ c_k k(k-1) + c_k \right] x^{k-2} = 0
\]

\[
= 2c_2 + 6c_3 x + \sum_{k=4}^{\infty} \left[ c_k (k-1) + c_{k-4} \right] x^{k-2} = 0
\]

c. \( c_2 = 0, \ c_3 = 0, \ c_0 \) and \( c_1 \) are free.

\( c_k k(k-1) + c_{k-4} = 0 \) for \( k = 4, 5, \ldots \) \( \Rightarrow \)

\[
c_k = - \frac{1}{k(k-1)} c_{k-4}, \quad k = 4, 5, \ldots
\]

Since \( c_2 = c_3 = 0 \) and \( c_6, \ c_{10}, \ldots \) and \( c_7, \ c_{11}, \ldots \) depend on \( c_2 \) and \( c_3 \),

6
\[ c_{4m+2} = 0, \text{ and } c_{4m+3} = 0, \text{ for } m = 0, 1, \ldots \]

<table>
<thead>
<tr>
<th>(k)</th>
<th>(c_4 = -\frac{1}{(4)(3)}c_0 = -\frac{1}{12}c_0)</th>
<th>(c_5 = -\frac{1}{(5)(4)}c_1 = -\frac{1}{20}c_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k = 8)</td>
<td>(c_8 = -\frac{1}{(8)(7)}c_4 = -\frac{1}{(8)(7)}\left(-\frac{1}{12}c_0\right) = \frac{1}{672}c_0)</td>
<td>(c_9 = -\frac{1}{(9)(8)}c_5 = -\frac{1}{(9)(8)}\left(-\frac{1}{20}c_1\right) = \frac{1}{1440}c_1)</td>
</tr>
</tbody>
</table>

\[ y = \sum_{m=0}^{\infty} c_m x^m = \sum_{m=0}^{\infty} c_{4m} x^{4m} + \sum_{m=0}^{\infty} c_{4m+1} x^{4m+1} \]
\[ = c_0 \left(1 - \frac{1}{12} x^4 + \frac{1}{672} x^8 + \ldots\right) + c_1 \left(x - \frac{1}{20} x^5 + \frac{1}{1440} x^9 + \ldots\right) \]

\[ y(0) = c_0 = 1, \quad y'(0) = c_1 = -1 \]

\[ y = \left(1 - \frac{1}{12} x^4 + \frac{1}{672} x^8 + \ldots\right) - \left(x - \frac{1}{20} x^5 + \frac{1}{1440} x^9 + \ldots\right) \]

**Example** Solve \(y'' + (\cos x)y = 0\) using the power series method and give the first 3 nonzero terms for each solution.

\(a_2(x) = 1 \neq 0\) and \(\cos x\) is analytic for all \(x\). So, the equation has no singular point.

\(a\) Let \(x_0 = 0\). \(\cos(x) = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \ldots\). Let \(y = \sum_{m=0}^{\infty} c_m x^m\). Then

\[ y' = \sum_{m=1}^{\infty} c_m m x^{m-1}, \quad y'' = \sum_{m=2}^{\infty} c_m m(m-1) x^{m-2}. \]

\[ \sum_{m=2}^{\infty} c_m m(m-1) x^{m-2} + \left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \ldots\right) \sum_{m=0}^{\infty} c_m x^m \]
\[ = \left(2(1) c_2 + 3(2) c_3 x + 4(3) c_4 x^2 + 5(4) c_5 x^3 + \ldots\right) \]
\[ + \left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \ldots\right) \left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots\right) \]
\[ = 2c_2 + c_0 + (6c_3 + c_1) x + \left(12c_4 + c_2 - \frac{1}{2} c_0\right) x^2 + \left(20c_5 + c_3 - \frac{1}{2} c_1\right) x^3 + \ldots = 0 \]

d.

\[ y = c_0 \left(1 - \frac{1}{2} x^2 + \frac{1}{12} x^4 + \ldots\right) + c_1 \left(x - \frac{1}{6} x^3 + \frac{1}{30} x^5 + \ldots\right) \]