1. Double Integrals:
Let \( f(x, y) \) be continuous in a region \( R \) which can be described as:
\[
R = \{(x, y) : g_1(x) \leq y \leq g_2(x), \quad x_1 \leq x \leq x_2\} \quad \text{or}
\[
R = \{(x, y) : h_1(y) \leq x \leq h_2(y), \quad y_1 \leq y \leq y_2\}
\]
\[
\int \int_{R(x,y)} f(x,y) \, dA = \int_{x_1}^{x_2} \int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \, dx \quad \text{or} \quad \int \int_{R(x,y)} f(x,y) \, dA = \int_{y_1}^{y_2} \int_{h_1(y)}^{h_2(y)} f(x,y) \, dx \, dy
\]
In polar coordinates: let
\[
R = \{(r, \theta) : g_1(\theta) \leq r \leq g_2(\theta), \quad \theta_1 \leq \theta \leq \theta_2\} \quad \text{or}
\[
R = \{(r, \theta) : h_1(r) \leq \theta \leq h_2(r), \quad r_1 \leq r \leq r_2\}
\]
\[
\int \int_{R(r,\theta)} f(r \cos \theta, r \sin \theta) \, dr \, d\theta = \int_{r_1}^{r_2} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta \quad \text{or} \quad \int \int_{R(r,\theta)} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta
\]
Remarks:
- When \( f(x, y) = 1 \), \( \int \int_{R(x,y)} f(x,y) \, dA \) is the area of \( R \).
- When \( f(x, y) \geq 0 \), \( \int \int_{R(x,y)} f(x,y) \, dA \) is the volume of the solid under the surface \( z = f(x,y) \) and above the region \( R(x,y) \).

Example Evaluate \( \int \int_{R} (y^2 - x) \, dA \) where \( R \) is a region bounded by the curves: \( y = x^2 + 1 \), and \( y = x + 3 \).

Sketch the graph of \( R \) : find the intersections of two curves:
\[
x^2 + 1 = x + 3, \quad x^2 - x - 2 = (x - 2)(x + 1) = 0, \quad x = 2, \quad x = -1.
\]

\[
\begin{align*}
\int \int_{R} (y^2 - x) \, dA &= \int_{-1}^{2} \int_{x^2+1}^{x+3} (y^2 - x) \, dy \, dx \\
&= \int_{-1}^{2} \left( \frac{1}{3} y^3 - xy \right)_{x^2+1}^{x+3} \, dx \\
&= \int_{-1}^{2} \left( \frac{1}{3} (x + 3)^3 - (x^2 + 1)^3 \right) - x(x + 3) + x(x^2 + 1) \, dx \\
&= \int_{-1}^{2} \left( \frac{4}{3} x^3 + 7x + \frac{26}{3} - \frac{1}{3} x^6 - x^4 \right) \, dx = \frac{2223}{70}
\end{align*}
\]
\[ \int \int_R (y^2 - x)\,dA = \int_1^2 \int_{\sqrt{y-1}}^{\sqrt{y+1}} (y^2 - x)\,dy\,dx + \int_2^5 \int_{\sqrt{y-1}}^{\sqrt{y+1}} (y^2 - x)\,dy\,dx \]
\[ = \int_1^2 \left( y^2x - \frac{1}{2}x^2 \right)_{\sqrt{y-1}}^{\sqrt{y+1}} \,dy + \int_2^5 \left( y^2x - \frac{1}{2}x^2 \right)_{\sqrt{y-1}}^{\sqrt{y+1}} \,dy \]
\[ = \int_1^2 \left( 2y^2\sqrt{y-1} - 0 \right) \,dy + \int_2^5 \left( y^2\left( \sqrt{y-1} - y + 3 \right) - \frac{1}{2}(y-1) + \frac{1}{2}(y-3)^2 \right) \,dy \]
\[ = \int_1^2 2y^2\sqrt{y-1} \,dy + \int_2^5 \left( y^2\sqrt{y-1} - y^3 + \frac{7}{2}y^2 - \frac{7}{2}y + 5 \right) \,dy = \frac{2223}{70} \]

**Example**  Evaluate \[ \int_0^4 \int_{y^2}^4 \cos(x^{3/2})\,dx\,dy \] algebraically (exactly).

Since we don’t know the antiderivative of \( \cos(x^{3/2}) \), we need to reverse the order of integration. Sketch the graph of \( R \) : \( 0 \leq y \leq 2, \ y^2 \leq x \leq 4 \)

![Graph of Region R](image)

\[ \int_0^4 \int_{y^2}^4 \cos(x^{3/2})\,dx\,dy = \int_0^4 \int_0^{\sqrt{x}} \cos(x^{3/2})\,dy\,dx = \int_0^4 \cos(x^{3/2}) \sqrt{x} \,dx = \int_0^4 \sqrt{x} \cos(x^{3/2})\,dx \]
\[ = \int_0^8 \frac{2}{3} \cos(u)\,du = \frac{2}{3} \sin 8 \]

**Example**  Evaluate \[ \int_0^2 \int_0^{\sqrt{8-x^2}} \frac{1}{5+x^2+y^2} \,dy\,dx \] algebraically.

Sketch the graph of \( R \) : \( 0 \leq x \leq 2, \ x \leq y \leq \sqrt{8-x^2} \)

![Graph of Region R](image)

\[ R : \ \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}, \ 0 \leq r \leq \sqrt{8} \]
\[ \int_0^2 \int_x^{\sqrt{8-x^2}} \frac{1}{5+x^2+y^2} \ dy \ dx = \int_{\pi/4}^{\pi/4} \int_0^{\sqrt{8}} \frac{1}{5+r^2} \ r \ dr \ d\theta = \int_{\pi/4}^{\pi/4} \frac{1}{2} \ ln(5+r^2) \bigg|_0^{\sqrt{8}} \ d\theta = \frac{1}{2} \ [ln(13) - ln(5)] \bigg( \frac{\pi}{2} - \frac{\pi}{4} \bigg) = \frac{\pi}{8} \ ln \bigg( \frac{13}{5} \bigg) \]

**Example**  
Find the volume of the solid bounded by  
\[ x^2 + y^2 = 1, \ x^2 + y^2 = 9, \ y \geq 0, \ z = 0, \ z = \sqrt{16 - x^2 - y^2} \]

Sketch the solid:

![Solid](image)

\[ V = \int_0^\pi \int_1^3 \sqrt{16-r^2} \ r \ dr \ d\theta = -\frac{\pi}{2} \frac{2}{3} \sqrt{(16-r^2)^3} \bigg|_1^3 = \frac{\pi}{3} \left( (15)^{3/2} - (7)^{3/2} \right) = \frac{1}{3} \pi \left( 15\sqrt{15} - 7\sqrt{7} \right) \]

2. **Green’s Theorem:**

**Positive Direction:**

The **positive direction** around a **simple closed curve** \( C \) is the direction a person must walk on \( C \), in order to keep the region \( R \) bounded by \( C \) to the left.

**Green’s Theorem:**

Suppose that the curve \( C \) is a **piecewise smooth simple closed curve** bounding a region \( R \) in a **positive direction**. If functions \( P, Q, \ \frac{\partial P}{\partial y} \) and \( \frac{\partial Q}{\partial x} \) are **continuous** on \( R \), then

\[ \oint_C P \ dx + Q \ dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \ dA \]

**Note:** Let \( \mathbf{F}(x,y) = \left[ P(x,y), \ Q(x,y), \ 0 \right] \), and \( d \mathbf{r} = \left[ dx, \ dy, \ dz \right] \). Since

\[ \text{curl} \ \mathbf{F} = \left[ 0, \ 0, \ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] \] and \( \text{curl} \ \mathbf{F} \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \),

\[ \oint_C P \ dx + Q \ dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \ dA \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl} \ \mathbf{F} \cdot \mathbf{k} \ dA \]

**Proof:** Assume that the region \( R \) can be represented as the following two ways.

**a.** \( R \ : \ a \leq x \leq b, \ g_1(x) \leq y \leq g_2(x) \) where \( g_1(b) = g_2(b) \) and \( g_1(a) = g_2(a) \), and \( C \) is the boundary of \( R \) around the positive direction, that is, from \( (a, g_1(a)) \) to \( (b, g_1(b)) \) along the curve \( y = g_1(x) \) and
then from \((b, g_2(b))\) to \((a, g_2(a))\) along the curve \(y = g_2(x)\).

\[
\int \int_R \frac{\partial P}{\partial y} \, dA = \int_a^b \left[\frac{g_2(x)}{g_1(x)} \frac{\partial P}{\partial y} \right] \, dy dx = \int_a^b P(x, y) \left[\frac{g_2(x)}{g_1(x)}\right] \, dx
\]

\[
= \int_a^b (P(x, g_2(x)) - P(x, g_1(x))) \, dx = -\int_a^b (P(x, g_1(x)) - P(x, g_2(x))) \, dx
\]

\[
= -\left(\int_a^b P(x, g_1(x)) \, dx + \int_b^a P(x, g_2(x)) \, dx\right) = -\oint_C P(x, y) \, dx
\]

**b.** \(R: c \leq y \leq d, \ h_1(y) \leq x \leq h_2(y)\) where \(h_1(d) = h_2(c)\) and \(h_1(c) = h_2(d)\), and \(C\) is the boundary of \(R\) around the positive direction, that is, from \((c, h_2(c))\) to \((d, h_2(d))\) along the curve \(x = h_2(y)\) and then from \((d, h_1(d))\) to \((c, h_1(c))\) along the curve \(x = h_1(y)\).

\[
\int \int_R \frac{\partial Q}{\partial x} \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} \frac{\partial Q}{\partial x} \, dx dy = \int_c^d Q(x, y) \left[\frac{h_2(y)}{h_1(y)}\right] \, dy
\]

\[
= \int_c^d (Q(h_2(y), y) - Q(h_1(y), y)) \, dy = \int_c^d Q(h_2(y), y) \, dy + \int_d^c Q(h_1(y), y) \, dy = \oint_C Q(x, y) \, dy
\]

Therefore,

\[
\int \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \, dA = \oint_C Q(x, y) \, dy - \oint_C P(x, y) \, dx = \oint_C P \, dx + Q \, dy
\]

**Example** Evaluate \(\oint_C (x^2 - y^2) \, dx + (2y - x) \, dy\) where \(C\) consists of the boundary of the region in the first quadrant that is bounded by \(y = x^2\) and \(y = x^3\) counterclockwise.

Sketch the region \(R\):

![Region R](image)

\[
P(x, y) = x^2 - y^2, \quad Q(x, y) = 2y - x, \quad \frac{\partial P}{\partial y} = -2y, \quad \frac{\partial Q}{\partial x} = -1
\]

\[
\oint_C (x^2 - y^2) \, dx + (2y - x) \, dy = \int_0^1 \int_{x^3}^{x^2} (-1 + 2y) \, dy \, dx = -\frac{11}{420}
\]

Check with the line integral:

\[
C_1: \overrightarrow{r}_1(x) = [x, x^3], \quad 0 \leq x \leq 1, \quad \overrightarrow{r}_1'(x) = [1, 3x^2]
\]

\[
C_2: \overrightarrow{r}_2(x) = [x, x^2], \quad x \text{ is from } 1 \text{ to } 0, \quad \overrightarrow{r}_2'(x) = [1, 2x];
\]

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\[ \oint_C (x^2 - y^2)dx + (2y - x)dy = \int_0^1 (x^2 - x^6)dx + \int_0^1 (2x^3 - x)(3x^2)dx + \int_0^1 (x^2 - x^4)dx + \int_0^1 (2x^2 - x)2xdx = -\frac{11}{420} \]

**Example**  Use Green’s Theorem to evaluate the line integral \( \oint_C (x^5 + 3y)dx + (2x - e^{y^3})dy \) where \( C \) is the circle: \((x - 1)^2 + (y - 5)^2 = 4\) counterclockwise.

![Sketch the graph of C](image)

**Example**  Evaluate the work done by the force vector \( \vec{F}(x,y) = [-16y + sin(x^2),
4e^y + 3x^2] \) along counterclockwise the curve \( C \) which is the boundary of: \( y = x, \ y = -x, \) and \( x^2 + y^2 = 1. \)

Sketch the graph of \( C = C_1 \cup C_2 \cup C_3 \)

\( C_1 : \ 0 \leq x \leq \frac{1}{\sqrt{2}}, \ y = x; \ C_2 : -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}, \ y = \sqrt{1-x^2}; \ C_3 : -\frac{1}{\sqrt{2}} \leq x \leq 0 \)
\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_3} \mathbf{F} \cdot d\mathbf{r} \]
\[ = \int_0^{\sqrt{2}} \left[-16x + \sin(x^2), 4e^x + 3x^2\right] \cdot [1, 1] \, dx \]
\[ - \int_{-0.7}^{0.7} \left[-16\sqrt{1-x^2} + \sin(x^2), 4e^{\sqrt{1-x^2}} + 3x^2\right] \cdot \left[1, \frac{-x}{\sqrt{1-x^2}}\right] \, dx \]
\[ + \int_0^1 \left[16x + \sin(x^2), 4e^{-x} + 3x^2\right] \cdot [1,-1] \, dx \]
\[ = 0.5817766 + 20.33484 - 8.35025 = 12.56637 \]

By Green’s Theorem:
\[ P(x,y) = -16y + \sin(x^2), \quad Q(x,y) = 4e^y + 3x^2, \quad \frac{\partial P}{\partial y} = -16, \quad \frac{\partial Q}{\partial x} = 6x, \quad \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} = 6x + 16 \]
\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{\pi/4}^{\pi/4} \int_0^1 (6\cos(\theta) + 16) \, r \, dr \, d\theta = 4\pi = 12.56637 \]

**Example** Evaluate \( \int_C \frac{-y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy \) where \( C \) is the boundary of the square \( R : -2 \leq x \leq 2, \ -2 \leq y \leq 2 \) counterclockwise.

Sketch the region \( R \):

![Sketch of the region](attachment:image.png)

**Region R**

Can we apply Green’s Theorem to this line integral? The answer is **NO** since both \( P(x,y) = \frac{-y}{x^2 + y^2} \), and \( Q(x,y) = \frac{x}{x^2 + y^2} \)

are **not defined** at \((0,0)\) and therefore they are not continuous everywhere on \( R \).

\[ \int_C \frac{-y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy = \int_{-2}^{2} \left(0 + \frac{2}{4 + y^2} \, dy\right) - \int_{-2}^{2} \left(\frac{-2}{x^2 + 4} \, dx + 0\right) \]
\[ - \int_{-2}^{2} \left(0 + \frac{-2}{x^2 + y^2} \, dy\right) + \int_{-2}^{2} \left(\frac{2}{x^2 + y^2} \, dx + 0\right) \]
\[ = 4 \int_{-2}^{2} \frac{2}{4 + y^2} \, dy = 2\pi \]

**Example** Evaluate \( \int_C \frac{-y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy \) where \( C \) is the boundary of the region \( R : -2 \leq x \leq 2, \ -2 \leq y \leq 2 \) and \( x^2 + y^2 \geq 1 \), counterclockwise.
Sketch the region $R$:

Since $P(x,y) = \frac{-y}{x^2 + y^2}$, and $Q(x,y) = \frac{x}{x^2 + y^2}$ are continuous on $R$, and

\[
\frac{\partial P}{\partial y} = \frac{-(x^2 + y^2) + y(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad \frac{\partial Q}{\partial x} = \frac{x^2 + y^2 - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}
\]

are also continuous on $R$, we can apply Green’s Theorem to evaluate the line integral.

\[
\oint_C \frac{-y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy = 0
\]