Directional Derivatives - (9.5)

1. **Gradient of a Scalar Function:**

   The Vector Differential Operator:
   \[
   \nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} \quad \text{or} \quad \nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} = \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right].
   \]

   Let \( f(x,y) \) and \( F(x,y,z) \) be differentiable functions. Then
   \[
   \nabla f(x,y) = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right], \quad \text{and} \quad \nabla F(x,y,z) = \left[ \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right].
   \]

   Vectors \( \nabla f(x,y) \) and \( \nabla F(x,y,z) \) are called gradient vectors of \( f \) and \( F \), respectively.

   Notations: \( \nabla f, \ \text{grad } f \).

**Example**

Let \( F(x,y,z) = 3e^{x^2+y^2} \cos \left( \frac{2x}{z} \right) \). Find \( \nabla F \) and \( \nabla F(\pi,1,2) \).

   \[
   \nabla F(x,y,z) = \left[ 6xe^{x^2+y^2} \cos \left( \frac{2x}{z} \right) - \frac{6x}{z} e^{x^2+y^2} \sin \left( \frac{2x}{z} \right), \frac{3x^2}{z} e^{x^2+y^2} \cos \left( \frac{2x}{z} \right), \frac{6x}{z^2} e^{x^2+y^2} \sin \left( \frac{2x}{z} \right) \right]
   \]

   \[
   \nabla F(\pi,1,2) = \left[ -6\pi e^{\pi^2}, -\frac{3\pi^2}{2} e^{\pi^2}, 0 \right]
   \]

   Use the Scientific Notebook: define \( F(x,y,z) \) first and then type \( \nabla F(x,y,z) \) and click on Evaluate. Here is what you get:

   \[
   \left( 6x \sqrt{y} e^{x^2+y^2} \cos 2 \frac{x}{z} - 6e^{x^2+y^2} \sin 2 \frac{x}{z}, \frac{3}{2} e^{x^2+y^2} \cos 2 \frac{x}{z}, 6e^{x^2+y^2} \left( \sin 2 \frac{x}{z} \right) \frac{x}{z^2} \right)
   \]

   To evaluate \( \nabla F(\pi,1,2) \), first define \( g(x,y,z) = \nabla F(x,y,z) \), that is,

   \[
   g(x,y,z) = \left( 6x \sqrt{y} e^{x^2+y^2} \cos 2 \frac{x}{z} - 6e^{x^2+y^2} \sin 2 \frac{x}{z}, \frac{3}{2} e^{x^2+y^2} \cos 2 \frac{x}{z}, 6e^{x^2+y^2} \left( \sin 2 \frac{x}{z} \right) \frac{x}{z^2} \right)
   \]

   and then type \( g(\pi,1,2) \) and click on Evaluate. Here is what you see:

   \[
   g(\pi,1,2) = \left( -6\pi e^{\pi^2}, -\frac{3\pi^2}{2} e^{\pi^2}, 0 \right)
   \]

2. **Directional Derivatives:**

   Let \( F(x,y,z) \) be differentiable. Partial derivatives \( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \) and \( \frac{\partial F}{\partial z} \) measure the rates of change of \( F(x,y,z) \) along the \( x \)-axis, \( y \)-axis and \( z \)-axis or in the direction \( \vec{u} = [1,0,0], \vec{v} = [0,1,0] \) and \( \vec{w} = [0,0,1] \), respectively. How to evaluate the rate of change of \( F \) in a given direction \( \vec{d} \)?

   **Definition**

   Let \( \vec{u} = [\cos(\theta), \sin(\theta)] \) be a unit vector in the \( xy \)-plane and let \( z = f(x,y) \) be a differentiable function. The directional derivative of \( f \) in the direction \( \vec{u} \) is defined as:

   \[
   D_{\vec{u}}f(x,y) = \lim_{h \to 0} \frac{f(x + h \cos(\theta), y + h \sin(\theta)) - f(x,y)}{h}
   \]

   provided the limit exists.

   Note that when \( \theta = 0 \), \( D_{\vec{u}}f(x,y) = \frac{\partial f}{\partial x} \), and when \( \theta = \pi \), \( D_{\vec{u}}f(x,y) = \frac{\partial f}{\partial y} \).

   Observe that the following.

   a.
\[
\frac{f(x + h \cos(\theta), y + h \sin(\theta)) - f(x, y)}{h} = \frac{f(x + h \cos(\theta), y + h \sin(\theta)) - f(x, y) + f(x + h \cos(\theta), y + h \sin(\theta)) - f(x, y)}{h}
\]

where the first term involves the change of \( f \) along the \( x \)-axis and the second term involves the change of \( f \) along the \( y \)-axis. By the Mean Value Theorem for differentiation \((g'(c) = \frac{g(b) - g(a)}{b-a})\), we know there exist \( x_1 \) between \( x \) and \( x + h \cos(\theta) \), and \( y_1 \) between \( y \) and \( y + h \sin(\theta) \) such that

\[
\frac{f(x + h \cos(\theta), y + h \sin(\theta)) - f(x, y)}{h} = \frac{\partial f}{\partial x} (x_1, y_1) (h \cos(\theta)) + \frac{\partial f}{\partial y} (x_1, y_1) (h \sin(\theta)).
\]

Then as \( h \to 0 \), \( x_1 \to x \) and \( y_1 \to y \), and

\[
D_u f(x, y) = \lim_{h \to 0} \left[ \frac{\frac{\partial f}{\partial x} (x_1, y_1) (h \cos(\theta)) + \frac{\partial f}{\partial y} (x_1, y_1) (h \sin(\theta))}{h} \right]
\]

\[
= \lim_{h \to 0} \left[ \frac{\partial f}{\partial x} (x_1, y_1) (\cos(\theta)) + \frac{\partial f}{\partial y} (x_1, y_1) (\sin(\theta)) \right]
\]

\[
= \frac{\partial f}{\partial x} (x_1, y) (\cos(\theta)) + \frac{\partial f}{\partial y} (x, y) (\sin(\theta)) = \nabla f(x, y) \cdot \vec{u}.
\]

In a similar way, we can derive a similar formula for computing \( D_\vec{v} F(x, y, z) = \nabla F(x, y, z) \cdot \vec{v} \).

b. Let \( \theta \) be the angle between vectors \( \vec{u} \) and \( \nabla F(x, y, z) \). Then

\[
D_\vec{u} F(x, y, z) = \nabla F(x, y, z) \cdot \vec{u} = ||\nabla F(x, y, z)|| ||\vec{u}|| \cos(\theta) = ||\nabla F(x, y, z)|| ||\vec{u}|| \cos(\theta).
\]

Since \(-1 \leq \cos(\theta) \leq 1\),

\[
-||\nabla F(x, y, z)|| \leq D_\vec{u} F(x, y, z) \leq ||\nabla F(x, y, z)||.
\]

The maximum value of the directional derivative is \( ||\nabla F(x, y, z)|| \) and it occurs when \( \vec{u} = \nabla F(x, y, z) \) and the maximum value of the direction derivative is \(-||\nabla F(x, y, z)|| \) and it occurs when \( \vec{u} = -\nabla F(x, y, z) \).

So, the gradient vector \( \nabla F \) points in the direction in which \( F \) increases most rapidly whereas \(-\nabla F \) points in the direction of the most rapid decrease of \( F \).

Example Let \( F(x, y, z) = \frac{x^2 - y^2}{z^2} \). Find the directional derivatives of \( F \) in the directions \( \vec{u} = \vec{i} - 2\vec{j} + 3\vec{k} \) and \( \vec{v} = \nabla F(x, y, z) \) at the point \((2, 4, -1)\).

\[
\nabla F(x, y, z) = \left[ \frac{2x}{z^2}, \frac{-2y}{z^2}, \frac{-2(x^2 - y^2)}{z^3} \right]. \quad \nabla F(2, 4, -1) = \left[ 4, \ -8, \ -24 \right]
\]

\[
D_\vec{u} F(2, 4, -1) = \left[ 4, \ -8, \ -24 \right] \cdot \frac{1}{\sqrt{1 + 4 + 9}} [1, -2, 3] = \frac{-26}{7} \sqrt{14} = -13.90
\]

\[
D_\vec{v} F(2, 4, -1) = \left[ 4, \ -8, \ -24 \right] \cdot \frac{1}{\sqrt{4^2 + 8^2 + 24^2}} \left[ 4, \ -8, \ -24 \right] = \sqrt{4^2 + 8^2 + 24^2} = 4 \sqrt{41} = 25.61
\]

Example Find a vector that gives the direction in which \( F(x, y, z) = \ln \left( \frac{xy}{z} \right) \) decreases most rapidly at \( \left( \frac{1}{2}, \ \frac{1}{6}, \ \frac{1}{3} \right) \).

The \( \vec{u} = -\nabla F \left( \frac{1}{2}, \ \frac{1}{6}, \ \frac{1}{3} \right) \) is a direction in which \( F(x, y, z) = \ln \left( \frac{xy}{z} \right) \) decreases most rapidly.
\[ F(x,y,z) = \ln(x) + \ln(y) - \ln(z), \quad \nabla F(x,y,z) = \left[ \frac{1}{x}, \frac{1}{y}, -\frac{1}{z} \right]. \]
\[ \nabla F\left( \frac{1}{2}, \frac{1}{6}, \frac{1}{3} \right) = [2, 6, -3], \quad \vec{u} = [-2, -6, 3]. \]

**Example** Suppose that \( \nabla f(a,b) = [2, -3] \). Find a unit vector \( \vec{u} \) so that (a) \( D_{\vec{u}} f(a,b) = 0 \), and (b) \( D_{\vec{u}} f(a,b) \) is a maximum.

a. Let \( \vec{u} = [\cos(\theta), \sin(\theta)] \). Then
\[ D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u} = 2\cos(\theta) - 3\sin(\theta) = 0 \Rightarrow 2\cos(\theta) = 3\sin(\theta), \quad \tan(\theta) = \frac{2}{3}, \]
\[ \theta = \tan^{-1}\left( \frac{2}{3} \right), \text{ and } \vec{u} = \left[ \cos\left( \tan^{-1}\left( \frac{2}{3} \right) \right), \sin\left( \tan^{-1}\left( \frac{2}{3} \right) \right) \right] = [0.83205, 0.5547]. \]

b. \[ \vec{u} = \frac{1}{|\nabla f(a,b)|} \nabla f(a,b) = \frac{1}{\sqrt{4 + 9}} [2, -3] = [0.5547, -0.83205] \]

**Example** The temperature \( T \) at a point \((x,y,z)\) in space is inversely proportional to the square of the distance from \((x,y,z)\) to the origin. It is known that \( T(0,0,1) = 500 \). Find the rate of change of \( T \) at \((2,3,3)\) in the direction of \((3,1,1)\). In which direction from \((2,3,3)\) does the temperature \( T \) increase most rapidly and what is the maximum rate of change of \( T \)?

The function \( T \) is
\[ T(x,y,z) = \frac{C}{x^2 + y^2 + z^2}, \quad T(0,0,1) = \frac{C}{1} = 500 \Rightarrow C = 500. \]
\[ \nabla T(x,y,z) = g(x,y,z) = \left[ \frac{-1000x}{(x^2 + y^2 + z^2)^2}, \frac{-1000y}{(x^2 + y^2 + z^2)^2}, \frac{-1000z}{(x^2 + y^2 + z^2)^2} \right] \]
\[ \nabla T(2,3,3) = g(2,3,3) = \left[ \frac{-500}{121}, \frac{-750}{121}, \frac{-750}{121} \right]. \]

The direction from \((2,3,3)\) in the direction of \((3,1,1)\) is \( \vec{u} = [1,-2,-2] \). So the rate of change is
\[ D_{\vec{u}} T(2,3,3) = \left[ \frac{-500}{121}, \frac{-750}{121}, \frac{-750}{121} \right] \cdot \frac{1}{\sqrt{1 + 4 + 4}} [1,-2,-2] = \frac{2500}{363}. \]

The temperature \( T \) increases most rapidly in the direction \( \vec{v} = \nabla T(2,3,3) \) and the maximum rate of change of \( T \) is
\[ D_{\vec{v}} T(2,3,3) = ||\nabla T(2,3,3)|| = ||\vec{v}|| = 9.69094. \]

**Example** Find a function \( f \) if possible such that \( \nabla f(x,y) = \left[ 2x\sqrt{y} e^{x^2 y^2}, \frac{x^2}{2\sqrt{y}} e^{x^2 y^2} + 2y\cos(y^2) \right] \).

\[ \frac{\partial f}{\partial x} = 2x\sqrt{y} e^{x^2 y^2} \Rightarrow f(x,y) = \int 2x\sqrt{y} e^{x^2 y^2} \, dx = e^{x^2 y^2} + C(y) \]
\[ \frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}} x^2 e^{x^2 y^2} + C'(y) = \frac{x^2}{2\sqrt{y}} e^{x^2 y^2} + 2y\cos(y^2) \]
\[ C'(y) = 2y\cos(y^2), \quad C(y) = \int 2y\cos(y^2) \, dy = \sin(y^2) + C \]
\[ f(x,y) = e^{x^2 y^2} + \sin(y^2) + C \]