1. Geometric Interpretation of the Gradient:
   a. Let \( z = f(x,y) \). Let \( P(a,b) \) be a point in the domain of \( z \). Consider the level curve 
      \[ C : f(x,y) = f(a,b) \].
   
   Let \( \vec{r}(t) = [x(t), y(t)] \) be a parametric representation of \( C \) and \( t_0 \) is a value at which 
   \[ \begin{cases} 
   x(t_0) = a \\
   y(t_0) = b 
   \end{cases} \]. Since 
   \[
   \frac{d}{dt} \left[ f(x,y) \right] = \frac{d}{dt} \left[ f(a,b) \right] = 0 \Rightarrow \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = 0, \quad \nabla f(x,y) \cdot \vec{r}'(t) = 0 
   \]
   \[
   \nabla f(a,b) \cdot \vec{r}'(t_0) = 0 
   \]
   \( \nabla f(a,b) \) is perpendicular to the level curve at the point \( P \).

b. Let \( w = F(x,y,z) \). Similarly, we can derive that \( \nabla F(a,b,c) \) is perpendicular to the level surface at \( P \).
   Let \( P(a,b,c) \) be a point in the domain of \( w \). Consider the level surface 
   \[ S : F(x,y,z) = F(a,b,c) \].
   
   Let 
   \[ \vec{r}(u,v) = [x(u,v), y(u,v), z(u,v)] \], for \( (u,v) \) in \( \mathbb{R} \) be a parametric representation of \( S \) and \( (u_0, v_0) \) is a pair of values at which 
   \[ \begin{cases} 
   x(u_0, v_0) = a \\
   y(u_0, v_0) = b \\
   z(u_0, v_0) = c 
   \end{cases} \]. Since 

\[
\frac{\partial}{\partial u} F(x(u,v),y(u,v),z(u,v)) = \frac{\partial}{\partial u} F(a,b,c) = 0 \Rightarrow \frac{\partial F}{\partial x} \frac{\partial}{\partial u} + \frac{\partial F}{\partial y} \frac{\partial}{\partial v} + \frac{\partial F}{\partial z} \frac{\partial}{\partial u} = 0
\]
\[
\nabla F(x,y,z) \cdot \frac{\partial F}{\partial u} (u,v) = 0, \quad \nabla F(a,b,c) \cdot \frac{\partial F}{\partial v} (u_0, v_0) = 0.
\]

Similarly,
\[
\nabla F(a,b,c) \cdot \frac{\partial F}{\partial v} (u_0, v_0) = 0.
\]

Since \(\frac{\partial F}{\partial u} (u_0, v_0)\) and \(\frac{\partial F}{\partial v} (u_0, v_0)\) are linearly independent and \(\nabla F(a,b,c)\) are perpendicular to both \(\frac{\partial F}{\partial u} (u_0, v_0)\) and \(\frac{\partial F}{\partial v} (u_0, v_0)\), \(\nabla F(a,b,c)\) is perpendicular to the plane spanned by \(\frac{\partial F}{\partial u} (u_0, v_0)\) and \(\frac{\partial F}{\partial v} (u_0, v_0)\) which is the plane tangent to the surface \(S\) at \(P(a,b,c)\).

Therefore, \(\nabla F(a,b,c)\) is perpendicular to the surface \(S\) at \(P(a,b,c)\).

2. **Tangent Plane and Normal line:**

Let \(P(x_0, y_0, z_0)\) be a point on the graph of \(F(x,y,z) = c\), where \(\nabla F\) is not \(\vec{0}\).

The **tangent plane** at this point is the plane that passes through the point \(P\) and is perpendicular to \(\nabla F\). So, the normal vector of the tangent plane is \(\nabla F = [F_x, F_y, F_z]\) and the equation of the plane is:
\[
F_x(x_0, y_0, z_0)(x-x_0) + F_y(x_0, y_0, z_0)(y-y_0) + F_z(x_0, y_0, z_0)(z-z_0) = 0.
\]

The **normal line** is the line that passes through the point \(P\) and is parallel to \(\nabla F\). So, the direction vector of the normal line is \(\nabla F = [F_x, F_y, F_z]\) and the parametric equations of the line are:
\[
\vec{r}(t) = [x_0, y_0, z_0] + t \nabla F(x_0, y_0, z_0)
\]
\[
= [x_0 + t F_x(x_0, y_0, z_0), y_0 + t F_y(x_0, y_0, z_0), z_0 + tF_z(x_0, y_0, z_0)]
\]
and symmetric equations of the line are:
\[
\frac{x-x_0}{F_x(x_0, y_0, z_0)} = \frac{y-y_0}{F_y(x_0, y_0, z_0)} = \frac{z-z_0}{F_z(x_0, y_0, z_0)}
\]

**Example** Find an equation of the tangent plane and parametric equations of the normal line to the graph of \(x^2 - 4y^2 + z^2 = 16\) at \((2,1,4)\).

\[
F(x,y,z) = x^2 - 4y^2 + z^2 - 16, \quad \nabla F(x,y,z) = [2x, -8y, 2z], \quad \nabla F(2,1,4) = [4, -8, 8]
\]

tangent plane : \(4(x-2) - 8(y-1) + 8(z-4) = 0\)
\[
x(t) = 2 + 4t
\]
\[
\text{normal line } : \vec{r}(t) = [2,1,4] + t[4, -8, 8] = [2 + 4t, 1 - 8t, 4 + 8t], \quad y(t) = 1 - 8t
\]
\[
z(t) = 4 + 8t
\]

**Example** Find an equation of the tangent plane and symmetric equations of the normal line to the
graph $z = \frac{1}{2}x^2 + \frac{1}{2}y^2 + 4$ at $(1,-1,5)$.

$F(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + 4 - z$, $\nabla F(x,y,z) = [x, y, -1]$, $\nabla F(1,-1,5) = [1,-1,-1]$

tangent plane: $(x-1) - (y + 1) - (z - 5) = 0$

normal line: $\frac{x - 1}{1} = \frac{y + 1}{-1} = \frac{z - 5}{-1}$