1. **Mean Value Theorem:**
   a. **Rolle’s Theorem:**
      Suppose that \( f \) is **continuous** on \([a, b]\) and is **differentiable** on \((a, b)\). If \( f(a) = f(b) \), then there exists a number \( c \) in \((a, b)\) such that \( f'(c) = 0 \).
   b. **Mean Value Theorem:**
      Suppose that \( f \) is **continuous** in \([a, b]\) and is **differentiable** on \((a, b)\). Then there exists a number \( c \) in \((a, b)\) such that
      \[
      f'(c) = \frac{f(b) - f(a)}{b - a}.
      \]

**Note that:**
   a. **Rolle’s Theorem** is a special case of the **Mean Value Theorem**.
   b. By Rolle’s Theorem, we know if \( f'(x) \neq 0 \) for all \( x \) in \((a, b)\), then \( f(a) \neq f(b) \). On the other hand, the condition \( f(a) \neq f(b) \) alone is not enough for us to determine if there exists a number \( c \) in \((a, b)\) such that \( f'(c) = 0 \).

**Graphically**, Rolle’s Theorem and the Mean Value Theorem can be described as follows.

![Graph](image)

\[
f(0) = f(2), \ f'(1) = 0
\]

c. When \( b \) is close to \( a \) (\(|b - a|\) is small), \( f'(c) \approx \frac{f(b) - f(a)}{b - a} \) and \( f'(x) \approx \frac{f(b) - f(a)}{b - a} \) for \( x \in (a, b) \).

**Example**  **First show that the equation** \( x - e^{-x} = 0 \) **has a solution in** \([0, 1]\). **Then Determine if the solution is unique.**

Let \( f(x) = x - e^{-x} \). Since \( f(0)f(1) = (0 - 1)\left(1 - \frac{1}{e}\right) = -0.632 < 0 \), by the Intermediate Value Theorem there exists a number \( c \) in \((0, 1)\) such that \( f(c) = 0 \). Therefore, the equation \( x - e^{-x} = 0 \) has a solution in \([0, 1]\). Now let us check to see if \( f'(x) = 0 \) for \( x \) in \((0, 1)\),

\[
f'(x) = 1 + e^{-x} > 0 \text{ for all } x.
\]

So, \( f'(x) \neq 0 \) for all \( x \) in \((0, 1)\) and therefore, \( f(x) = 0 \) only once and the solution is unique.
Example  The graph of \( f(x) = x \cos(x + 0.2) \) for \( x \) in \([-2, 2]\) is given below. Find graphically all possible \( c \) in \((-2, 2)\) satisfying the conclusion given in the Mean Value Theorem. Approximate \( f'(c) \) for each \( c \).

Approximately, \( c_1 = -1.19 \) and \( c_2 = 0.89 \).

\[
\begin{align*}
\frac{f'(c_1)}{2} & \approx \frac{f(-1) - f(-1.5)}{-1 + 1.5} = \frac{-0.697 - (-0.401)}{0.5} = -0.592 \\
\frac{f'(c_2)}{2} & \approx \frac{f(1) - f(0.5)}{1 - 0.5} = \frac{0.362 - 0.382}{0.5} = -0.04 \\
\end{align*}
\]

Comparison:
\[
\begin{align*}
f'(c_1) & \approx -0.446, \quad f'(c_2) = -0.327 \\
\end{align*}
\]

2. Fixed-Point of a Function:
A number \( p \) is said to be a **fixed point** of a function \( g(x) \) if \( g(p) = p \). Graphically, a function has a fixed point at \( x = p \) if its graph \( y = g(x) \) and the line \( y = x \) intersect at \( x = p \).

\[
e^{-p} = p, \text{ when } p \approx 0.58
\]

Some functions may have more than one fixed points and some functions may not have a fixed point. For example, the function in (i) has **no fixed point** and the function in (ii) has **infinitely many fixed points**.
Algebraically, we solve the equation $g(x) = x$ (or $g(x) - x = 0$) to determine if a function has any fixed point over a given interval.

**Example** Find all fixed points of $g_1(x) = x^2 + 1$ and $g_2(x) = x + \cos(x)$ if they exist.

a. Set $g_1(x) = x: x^2 + 1 = x$, $x^2 - x + 1 = 0$. Using the quadratic formula:

$$x = \frac{1 \pm \sqrt{1 - 4(1)}}{2} = \frac{1 \pm \sqrt{-3}}{2}$$

no real solution.

So, $g_1(x)$ has no fixed point for $-\infty < x < \infty$.

b. Set $g_2(x) = x: x + \cos(x) = x$, $\cos(x) = 0$, $x = \pm \frac{2n-1}{2}\pi$, $n = 1, 2, 3, \ldots$

So, $g_2(x)$ has infinitely many fixed points for $-\infty < x < \infty$.

### 3. Existence and Uniqueness of a Fixed Point:

Let $g$ be continuous on $[a, b]$.

i. If $a \leq g(x) \leq b$ for all $x$ in $[a, b]$, then $g(x)$ has a fixed point $p$ in $[a, b]$.

ii. If, in addition, $g'(x)$ exists on $(a, b)$ and there exists a constant $0 < K < 1$ such that

$$|g'(x)| \leq K$$

for all $x$ in $(a, b)$,

then $p$ is unique.

**Note that:**

a. Both conditions: $a \leq g(x) \leq b$ for all $x$ in $[a, b]$ and $|g'(x)| \leq K$ for all $x$ in $(a, b)$ are **sufficient conditions**. So, in the case where the condition in i. does not hold, it is possible that $g(x)$ has a fixed point; and in the case where the condition in ii. is not satisfied, it is also possible the fixed point of $g(x)$ is unique.

b. Because $g'(x)$ is the slope of the tangent line to the curve $y = g(x)$ at $x$, $|g'(x)| \leq K < 1$ means that the graph of $g(x)$ does not grow as faster than $y = x$ and not slower than $y = -x$.

**Proof:**

i. If $g(a) = a$ or $g(b) = b$, then $p = a$ or $p = b$ and $g$ has a fixed point. Now let $g(a) > a$ and $g(b) < b$, and let $h(x) = g(x) - x$. Since $g(x)$ is continuous on $[a, b]$, $h(x)$ is continuous on $[a, b]$. Observe that $h(a) = g(a) - a > 0$ and $h(b) = g(b) - b < 0$. So, by the Intermediate Value Theorem, we know there exists a number $c$ in $(a, b)$ such that $h(c) = 0$, that is

$$g(c) - c = 0$$ or $$g(c) = c.$$
So, \( c \) is a fixed point of \( g \) in \([a, b]\).

ii. Now let also \( |g'(x)| \leq K \) for all \( x \) in \((a, b)\) where \( 0 < K < 1 \). Suppose that \( g(x) \) has two fixed points, say \( p < q \) in \([a, b]\). Then by the Mean Value Theorem, we know there exists a point \( c \) in \((q, p)\) such that

\[
g'(c) = \frac{g(p) - g(q)}{p - q}.
\]

Since \( g(p) = p \) and \( g(q) = q \),

\[
\frac{g(p) - g(q)}{p - q} = \frac{p - q}{p - q} = 1.
\]

So, \( g'(c) = 1 \), this contradicts the given condition \( |g'(x)| < 1 \) for all \( x \) in \((a, b)\). So, \( g \) cannot have two fixed points in \((a, b)\).

**Example** Let \( g(x) = \frac{1}{3}(x^2 - 1) \) for \( x \) in \([-1, 1]\). Determine if \( g \) has a fixed point in \([-1, 1]\). If so, determine if the fixed point is unique.

Check:

i. Observe that \( g_{\text{min}} = g(0) = -\frac{1}{3} \geq -1 \) and \( g_{\text{max}} = g(1) = g(-1) = 0 \leq 1 \). Since \(-1 \leq g(x) \leq 1\) for all \( x \) in \([-1, 1]\), \( g \) has a fixed point in \([-1, 1]\).

ii. Compute \( g'(x) = \frac{2}{3}x \). Since \( |g'(x)| = \frac{2}{3}|x| \leq \frac{2}{3} < 1 \), \( g \) has a unique fixed point in \([-1, 1]\).

For \( g(x) \), we can solve its fixed point \( p \) algebraically.

\[
g(x) = x \Rightarrow \frac{1}{3}(x^2 - 1) = x \Rightarrow x^2 - 3x - 1 = 0 \Rightarrow x = \frac{3 \pm \sqrt{9 - 4(-1)}}{2} = \frac{3 \pm \sqrt{13}}{2}
\]

Since \( \frac{3 + \sqrt{13}}{2} > 1 \), \( p = \frac{3 - \sqrt{13}}{2} = -0.302776 \) is a unique fixed point in \([-1, 1]\).

Check the graph of \( g(x) \):

\[
y = \frac{1}{3}(x^2 - 1), \text{---} \ y = x, \ y =
\]

**Example** Let \( g(x) = 3^{-x} \) for \( x \) in \([0, 1]\). Determine if \( g \) has a fixed point in \([0, 1]\). If so, determine if the fixed point is unique.

Check:

i. Observe that \( g_{\text{min}} = g(1) = 3^{-1} > 0 \), and \( g_{\text{max}} = g(0) = 1 \). Since \( 0 \leq g(x) \leq 1 \), \( g(x) \) has a fixed point in \([0, 1]\).

ii. Compute \( g'(x) = -3^{-x} \ln 3 \). Since \( |g'(x)| = 3^{-x} \ln 3 \), there is no conclusion about the uniqueness.

From the graph of \( g \) below, we can see that \( g \) has a unique fixed point \( p \approx 0.55 \) in \([-1, 1]\). But we
cannot solve $p$ algebraically. How can solve a fixed point numerically?

$$y = 3^{-x}, \ x \in [0, 1]$$

4. The Fixed-Point Iteration:

It is an algorithm to find a fixed-point of a function over an interval assuming the fixed point is unique.

**Algorithm:** Given $g(x)$, and $[a, b]$, choose $p_0$ in $[a, b]$ and compute $p_1, p_2, \ldots$, as follows:

$$p_n = g(p_{n-1}) \quad \text{for } n = 1, 2, \ldots$$

Implement the algorithm in a programming language which does the following:

- Input $g(x)$, interval $[a, b]$, $p_0$ in $[a, b]$, $\epsilon$ and $K_{max}$, and compute $p_n = g(p_{n-1})$ for $n = 1, 2, \ldots$. The program terminates if
  - $|p_n - p_{n-1}| < \epsilon$ and then $p \approx p_n$; or
  - $p_n > b$ or $p_n < a$, and the program fails; or
  - $n = K_{max}$.

The following two examples show graphically how the Fixed-Point Iteration works.

Clearly, the Fixed-Point iteration finds the fixed point $p$ in (i) and diverges in (ii).

5. Convergence and the Rate of Convergence:
Questions: Assume that \( g \) has a unique fixed point \( p \) in \([a, b]\) and \( p_0 \) is in \([a, b]\). Let \( p_n = g(p_{n-1}), \ n = 1,2, \ldots \).

i. Under what condition(s), does \( p_n \) converge to \( p \)?

ii. If \( \lim_{n \to \infty} p_n = p \), what is the rate of converge?

**Fixed-Point Theorem:**

Let \( g \) be continuous on \([a, b]\) and \( a \leq g(x) \leq b \). Suppose that \( g'(x) \) exists for all \( x \) in \((a, b)\), and

\[
|g'(x)| \leq K \text{ for all } x \text{ in } (a, b) \text{ where } 0 < K < 1.
\]

Then \( \lim_{n \to \infty} p_n = p \) for any \( p_0 \) in \([a, b]\), and

\[
|p_n - p| \leq K^n \max\{p_0 - a, \ b - p_0\} \text{ and } |p_n - p| \leq \frac{K^n}{1-K} \left| p_1 - p_0 \right|, \text{ for all } n = 1,2, \ldots.
\]

**Proof:** Let \( p_0 \) be in \([a, b]\) and \( \{p_n\} \) be generated by the Fixed-Point Iteration. Observe that

\[
|p_n - p| = \left| g(p_{n-1}) - g(p) \right|
\]

By the Mean Value Theorem, we know there exists a number \( c \) in \((a, b)\) such that

\[
g(p_{n-1}) - g(p) = g'(c)(p_{n-1} - p).
\]

Since \( |g'(x)| < 1 \) for all \( x \) in \((a, b)\),

\[
|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(c)(p_{n-1} - p)| = |g'(c)| \left| p_{n-1} - p \right| \leq K \left| p_{n-1} - p \right|
\]

\[
\leq K \left| p_{n-2} - p \right| = K^2 \left| p_{n-2} - p \right| \ldots
\]

\[
\leq K^n |p_0 - p|
\]

\[
0 < \lim_{n \to \infty} |p_n - p| \leq \lim_{n \to \infty} K^n |p_0 - p| = 0.
\]

Therefore, \( \lim_{n \to \infty} p_n = p \).

Since \( |p_0 - \mathbf{p}| \leq |p_0 - a| \) or \( |p_0 - \mathbf{p}| \leq |b - p_0| \),

\[
|p_n - \mathbf{p}| \leq K^n |p_0 - \mathbf{p}| \leq K^n \max\{p_0 - a, b - p_0\}.
\]

Observe that

\[
|p_{n+1} - p_n| = |g(p_n) - g(p_{n-1})| \leq K \left| p_n - p_{n-1} \right| = K \left| g(p_{n-1}) - g(p_{n-2}) \right|
\]

\[
\leq K^2 \left| g(p_{n-2}) - g(p_{n-3}) \right| \leq \ldots
\]

\[
\leq K^n |p_1 - p_0|
\]

So for any \( m > n \geq 1 \),

\[
|p_m - p_n| = |(p_m - p_{m-1}) + (p_{m-1} - p_{m-2}) + \ldots + (p_{n+1} - p_n)|
\]

\[
\leq |p_m - p_{m-1}| + |p_{m-1} - p_{m-2}| + \ldots + |p_{n+1} - p_n|
\]

\[
\leq K^{m-1} |p_1 - p_0| + K^{m-2} |p_1 - p_0| + \ldots + K^n |p_1 - p_0|
\]

\[
= K^n (K^{m-n-1} + K^{m-n-2} + \ldots + 1) |p_1 - p_0|
\]

\[
= K^n \left( \frac{1 - K^{m-n}}{1-K} \right) |p_1 - p_0|
\]

Since \( \lim_{m \to \infty} p_m = p \),
\[ |p - p_n| = \lim_{m \to \infty} |p_m - p_n| \leq \lim_{m \to \infty} K^n \left( \frac{1 - K^{m-n}}{1 - K} \right) |p_1 - p_0| = \frac{K^n}{1 - K} |p_1 - p_0| \]

Note that:

a. Rate of convergence: 
\[ |p_n - p| \leq \frac{K^n}{1 - K} |p_1 - p_0|, \text{ for all } n = 1, 2, \ldots \] implies that 
\[ p_n \to p \text{ with the rate of convergence of } O(K^n), \text{ i.e., } p_n = p + O(K^n). \]

b. Order of convergence: From 
\[ |p_n - p| = |g'(c_{n-1})||p_{n-1} - p| \leq K |p_{n-1} - p|, \] we have 
\[ \frac{|p_n - p|}{|p_{n-1} - p|} \leq K. \]

Hence, \( \{p_n\} \) converges to \( p \) linearly (\( \alpha = 1 \)) with an asymptote error constant \( K \).

c. The smallest possible number of iterations: For a given \( \varepsilon \), we can estimate the number \( N \) of iterations needed to approximate \( p \) by \( p_N \). That is, find \( N \) such that
\[ \frac{K^n}{1 - K} |p_1 - p_0| < \frac{K^n}{1 - K} |b - a| < \varepsilon \Rightarrow K^n < \varepsilon \left( \frac{1 - K}{b - a} \right) \Rightarrow N \ln(K) < \ln \left( \varepsilon \left( \frac{1 - K}{b - a} \right) \right) \]

Since \( 0 < K < 1 \), \( \ln(K) < 0 \). So, \( N > \frac{\ln \varepsilon + \ln \left( \frac{1 - K}{b - a} \right)}{\ln(K)} \).

Example

Determine whether or not the function has a fixed point in the given interval. If so, determine if the Fixed-point Iteration will converge to the fixed point. In the case when it converges, estimate the number of iterations possibly needed to approximate the fixed point within \( 10^{-5} \).

(i) \( g(x) = \frac{1}{3}(2 - e^x + x^2), \ [0, 1] \)

(ii) \( g(x) = \frac{1}{2}(10 - x^3)^{1/2}, \ [0, 2] \)

(i) Check the range of \( g \) : From the graph of \( g(x) \) on \([0, 1]\), we have \( 0 \leq g(x) \leq 1 \).

So, \( g(x) = \frac{1}{3}(2 - e^x + x^2) \) has a fixed point in \([0, 1]\).

Check the maximum value of \( |g'(x)| \) : 
\[ g'(x) = \frac{1}{3}(-e^x + 2x). \] From the graph of \(|g'(x)|\),
\[ |g'(x)| \leq |g'(0)| = \left| \frac{1}{3}(-1) \right| = \frac{1}{3} = K < 1 \text{ for all } x \text{ in } [0, 1]. \]

So, \( g(x) \) has a unique fixed-point in \([0, 1]\) and the sequence \( \{p_n\} \) generated by the Fixed-Point Iteration converges to \( p \) linearly (\( \alpha = 1 \)) with an asymptote error constant \( K \).
Iteration converges to $p$. 
Estimate the number $N$ of iterations:

$$N > \frac{\ln 10^{-5} + \ln \left(1 - \frac{1}{10}\right)}{\ln \left(\frac{1}{10}\right)} = 10.8486, \quad N = 11.$$ 

Use the Fixed-Point Iteration to solve the fixed point in $[0,1]$ and $p \approx p_n$ where $|p_n - p_{n-1}| < 10^{-5}$.

$p_0 = 0$

0.33333333333333
0.23849956200834
0.26251296366787
0.25623991092001
0.25786540708179
0.25744331555362
0.2575285995622
0.2575242613046
0.25753180626754

$n = 9$. 

b. Check the range of $g(x)$: Observe that

$$g_{\text{min}} = g(2) = \frac{1}{2} \sqrt{10-8} = \frac{\sqrt{2}}{2} \geq 0, \quad g_{\text{max}} = g(0) = \frac{1}{2} \sqrt{10} \leq 2.$$ 

So, $0 \leq g(x) \leq 2$ for all $x$ in $[0,2]$. Hence, $g(x)$ has a fixed point in $[0,2]$.

Check the maximum value of $|g'(x)|$:

$$g'(x) = \frac{1}{2} \frac{-3x^2}{\sqrt{10-x^3}}, \quad |g'(x)| = \frac{3x^2}{2\sqrt{10-x^3}}.$$ 

From the graph of $|g'(x)|$:

$$y = |g'(x)| = \left| \frac{1}{2} \frac{-3x^2}{\sqrt{10-x^3}} \right|$$

$|g'(x)| > 1$ for some $x$ in $[0,2]$. So we cannot conclude the sequence $\{p_n\}$ generated by the Fixed-Point Iteration converges to $p$. 

Use the Fixed-Point Iteration to solve the fixed point in $[0,1]$ and $p \approx p_n$ where $|p_n - p_{n-1}| < 10^{-5}$. 

8
6. Fixed-Point Iteration for Solving The Equation: \( f(x) = 0 \)

Let \( x^* \) be a solution of the equation \( f(x) = 0 \). To solve \( x^* \) using the Fixed-Point Iteration, a function \( g \) needs to be defined first such that \( x^* \) is a fixed point of \( g \), that is, \( x^* = g(x^*) \).

**Example** Consider solving \( x^3 + x + 1 = 0 \). Find an interval \([a, b]\) on which the equation has a solution. Find a function \( g \) such that the fixed point of \( g \) is the solution of the equation: \( f(x) = 0 \). Determine if the sequence \( \{p_n\} \) generated by the Fixed-Point Iteration with the function \( g \).

Consider \([a, b] = [-1, 0]\). Since \( f(-1)f(0) = (-1)(1) < 0 \), the equation has a solution in \([-1, 0]\).

1. A naive choice of \( g \): since \( x = -1 - x^3 \), we can let \( g(x) = -1 - x^3 \).
   - Check the range of \( g \): \( g_{\text{min}} = g(0) = -1 \) and \( g_{\text{max}} = g(-1) = 0 \), so, \(-1 \leq g(x) \leq 0 \) and \( g \) has a fixed point in \([-1, 0]\).
   - Check the maximum value of \( |g'(x)| \): \( g'(x) = -3x^2 \), \( |g'(x)| = 3x^2 > 1 \) for some \( x \) in \([-1, 0]\).
   - So, it is not certain by the Fixed-Point Theorem if \( \{p_n\} \) converges to \( p \).
   - Observe that \( \{p_n\}_{n=0} = \{0, -1, 0, -1, \ldots\} \).

2. Rewrite the equation \( x^3 + x + 1 = 0 \) as \( x^3 = -1 - x \), \( x = -\sqrt[3]{-x + 1} \). Let \( g(x) = -\sqrt[3]{x + 1} \).
   - Check the range of \( g \): \( g_{\text{min}} = g(0) = -1 \), \( g_{\text{max}} = g(-1) = 0 \), so, \(-1 \leq g(x) \leq 0 \) and \( g \) has a fixed point in \([-1, 0]\).
   - Check the maximum value of \( |g'(x)| \): \( g'(x) = -\frac{1}{3} \frac{1}{\sqrt[3]{(x+1)^2}} \). Since \( g'(x) \) is not defined at \( x = -1 \), \( g'(x) \) is unbounded. So, it is not certain if \( \{p_n\} \) converges to \( p \).
   - Observe that \( \{p_n\}_{n=0} = \{0, -1, 0, -1, \ldots\} \).

3. Rewrite the equation \( x^3 + x + 1 = 0 \) as \( x^3 + x = -1 \) and then \( x(x^2 + 1) = -1 \) or \( x = \frac{-1}{x^2 + 1} \). Let \( g(x) = \frac{-1}{x^2 + 1} \).
   - Check the range of \( g \): \( g_{\text{min}} = g(0) = -1 \), and \( g_{\text{max}} = g(-1) = -\frac{1}{2} \), so, \(-1 \leq g(x) \leq 0 \) and \( g \) has a fixed point in \([-1, 0]\).
Check the maximum value of $|g'(x)|$:

| $g'(x)$ | $|g'(x)|$ |
|---------|---------|
| $-\frac{2x}{(x^2+1)^2}$ | $\frac{2|x|}{(x^2+1)^2}$ |

From the graph of $|g'(x)|$ above, we see $|g'(x)| \leq 0.7 = K < 1$. So, $p$ is unique in $[-1, 0]$ and $\{p_n\}$ converges to $p$.

Estimate the number $N$ of iterations needed:

$$N > \frac{\ln 10^{-5} + \ln(1 - 0.7)}{\ln(0.7)} = 35.654, \quad N = 36.$$  

Use the Fixed-Point Iteration to solve the fixed point in $[0, 1]$ and $p \approx p_n$ where

$$|p_n - p_{n-1}| < 10^{-5}.$$  

$n = 27$ and $p_{27} = -0.68232442571947$.

**Example** Show that each of the following functions has a fixed point at $p$ precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.

a. $g(x) = (3 + x - 2x^2)^{1/4}$  
b. $g(x) = \frac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1}$

a. Set $x^4 + 2x^2 - x - 3 = 0$. Then

$$x^4 = 3 + x - 2x^2 \implies x = (3 + x - 2x^2)^{1/4}.$$  

b. Check if $x = \frac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1}$. Then

$$x - \frac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1} = \frac{4x^4 + 4x^2 - x - (3x^4 + 2x^2 + 3)}{4x^3 + 4x - 1} = \frac{x^4 + 2x^2 - x - 3}{4x^3 + 4x - 1} = 0.$$  

So, $x = \frac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1}$.

**Example** The following four methods are proposed to compute $7^{1/5}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

(i) $p_n = \left(1 + \frac{7 - p_{n-1}^3}{p_{n-1}^2}\right)^{1/2}$  
(ii) $p_n = p_{n-1} - \frac{p_{n-1}^5 - 7}{p_{n-1}^2}$

(iii) $p_n = p_{n-1} - \frac{p_{n-1}^5 - 7}{5p_{n-1}^4}$  
(iv) $p_n = p_{n-1} - \frac{p_{n-1}^5 - 7}{12}$

The function $g$ for each iteration is:
(i) \( g_1(x) = \left(1 + \frac{7-x^3}{x^2}\right)^{1/2} = \left(1 + \frac{7}{x^2} - x\right)^{1/2} \)

(ii) \( g_2(x) = x - \frac{x^5 - 7}{x^2} = x - x^3 + \frac{7}{x^2} \)

(iii) \( g_3(x) = x - \frac{x^5 - 7}{5x^4} = x - \frac{1}{5}x^2 + \frac{7}{5x^4} \)

(iv) \( g_4(x) = x - \frac{x^5 - 7}{12} = x - x^3 + \frac{7}{12} \)

All four functions have a fixed point on \([a, b]\). Compute \( g'(x) \):

(i) \( g'_1(x) = \frac{1}{2} \left(1 + \frac{7}{x^2} - x\right)^{-1/2} \left(-\frac{14}{x^3} - 1\right) \)

(ii) \( g'_2(x) = 1 - 3x^2 - \frac{14}{x^3} \)

(iii) \( g'_3(x) = \frac{4}{5} - \frac{28}{5x^5} \)

(iv) \( g'_4(x) = 1 - \frac{5}{12}x^4 \)

Check the value of \( |g'(x)| \) at \( p = 7^{1/5} \):

\[ |g'_1(p)| = \left| \frac{1}{2} \left(1 + \frac{7}{p^2} - p\right)^{-1/2} \left(-\frac{14}{p^3} - 1\right) \right| = 1.61828 > 1 \]

\[ |g'_2(p)| = \left| 1 - 3p^2 - \frac{14}{p^3} \right| = 9.88953 > 1 \]

\[ |g'_3(p)| = \left| \frac{4}{5} - \frac{28}{5p^5} \right| = 0 < 1 \]

\[ |g'_4(p)| = \left| 1 - \frac{5}{12}p^4 \right| = 0.976365 < 1 \]

So, the sequences \( \{p_n\} \) generated by the Fixed-Point Iteration using \( g_1 \) and \( g_2 \) do not converge. The sequences \( \{p_n\} \) generated by the Fixed-Point Iteration using \( g_3 \) and \( g_4 \) converge and the third sequence converges faster than the 4th one. The testing results show that:

- Using \( g_3(x) \), \( 7^{1/5} \approx p_7 = 1.47577316159456 \)
- Using \( g_4(x) \), \( 7^{1/5} \approx p_{355} = 1.47577807080213 \)
Example  Show that if \( A \) is any positive number, then the sequence defined by

\[
x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}}, \text{ for } n \geq 1,
\]

converges to \( \sqrt{A} \) whenever \( x_0 > 0 \).

Let \( \lim_{n \to \infty} x_n = x \). Then

\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} \left( \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}} \right) \iff x = \frac{1}{2}x + \frac{A}{2x} \iff \\
\frac{1}{2}x = \frac{A}{2x} \iff x^2 = A \iff x = \sqrt{A}.
\]