3.4 - Cubic Spline Interpolation

Cubic Spline Approximation:

Problem: Given $n+1$ pairs of data points $(x_i,\ y_i),\ i=0,1,\ldots,n,$ find a piecewise-cubic polynomial $S(x)$

$$S(x)=\begin{cases} S_0(x)=a_0+b_0(x-x_i)+c_0(x-x_0)^2+d_0(x-x_0)^3 & \text{if } x_0\le x\le x_1\\ S_1(x)=a_1+b_1(x-x_1)+c_1(x-x_1)^2+d_1(x-x_1)^3 & \text{if } x_1\le x\le x_2\\ : & :\\ S_{n-1}(x)=a_{n-1}+b_{n-1}(x-x_{n-1})+c_{n-1}(x-x_{n-1})^2+d_{n-1}(x-x_{n-1})^3 & \text{if } x_{n-1}\le x\le x_n\end{cases}$$

and $S(x_i)=y_i,\ i=0,1,\ldots,n.$

Cubic Splines:

$$S_i(x)=a_i+b_i(x-x_i)+c_i(x-x_i)^2+d_i(x-x_i)^3,\ i=0,1,\ldots,n-1.$$

Conditions:

(1) $S_i(x_i)=y_i,\ i=0,1,2,..,n$ - interpolating data $(x_i,\ y_i)$

(2) $S_i(x_{i+1})=S_{i+1}(x_{i+1}),\ i=0,1,\ldots,n-2$ – continuity at interior points

(3) $S_i{}'(x_{i+1})=S_{i+1}{}'(x_{i+1}),\ i=0,1,\ldots,n-2$ - continuous slope at interior points

(4) $S_i{}''(x_{i+1})=S''_{i+1}(x_{i+1}),\ i=0,1,\ldots,n-2$ - continuous curvature at interior points

(5) $S''_0(x_0)=0,$ and $S''_{n-1}(x_n)=0$ – **free spline or natural spline**

$S'_0(x_0)=\alpha$ and $S'_{n-1}(x_n)=\beta$ - **clamped spline**

Totally we have $4n$ unknowns: $a_i,\ b_i,\ c_i,$ and $d_i,\ i=0,1,\ldots,n-1$ and there are $n+1$ equation in (1), $n-1$ equations in (2), $n-1$ equations in (3) $n-1$ conditions in (4) and 2 equations in (5). So, again we can solve $a_i,\ b_i,\ c_i$ and d_i uniquely.

From (1): $S_i(x_i)=y_i$, we have

$$S_i(x_i)=a_i=y_i,\ i=0,1,2,..,n-1,$$

and $S_{n-1}(x_n)=y_n,$ we have an equation in $b_{n-1},\ c_{n-1}$ and d_{n-1}:

$$y_{n-1}+b_{n-1}(x_n-x_{n-1})+c_{n-1}(x_n-x_{n-1})^2+d_{n-1}(x_n-x_{n-1})^3=y_n. \qquad (1.1)$$

From (2): $S_i(x_{i+1})=S_{i+1}(x_{i+1}),\ i=0,1,\ldots,n-2,$ we have $n-1$ equations in $b_i,\ c_i$ and d_i,

$$y_i+b_i(x_{i+1}-x_i)+c_i(x_{i+1}-x_i)^2+d_i(x_{i+1}-x_i)^3=y_{i+1},\ i=0,1,\ldots,n-2. \qquad (2.1)$$

$S_i'(x)=b_i+2c_i(x-x_i)+3d_i(x-x_i)^2$. From (3): $S_i{}'(x_{i+1})=S_{i+1}{}'(x_{i+1}),\ i=0,1,\ldots,n-2,$ we have $n-1$ equations in $b_i,\ c_i$ and d_i:

$$b_i+c_i(x_{i+1}-x_i)+3d_i(x_{i+1}-x_i)^2=b_{i+1},\ i=0,1,\ldots,\ n-2. \qquad (3.1)$$

$S''_i(x)=6d_i(x-x_i)+2c_i$. From (4): $S_i{}''(x_{i+1})=S_{i+1}{}''(x_{i+1}),\ i=0,1,\ldots,n-2,$ we have $n-1$ equations in c_i and d_i:

$$2c_i+6d_i(x_{i+1}-x_i)=2c_{i+1},\ i=0,1,\ldots,n-2. \qquad (4.1)$$

For a **free spline or natural spline**, from conditions: $S''_0(x_0)=0,$ and $S''_{n-1}(x_n)=0,$ we have an equation in c_0 and an equation in c_{n-1} and d_{n-1}:

$$2c_0=0,\ \text{and}\ 6d_{n-1}(x_n-x_{n-1})+2c_{n-1}=0. \qquad (5.1)$$

For a **clamped spline**, from conditions: $S'_0(x_0)=\alpha$ and $S'_n(x_n)=\beta$, we have a condition in b_0 and an equation in $b_{n-1},\ c_{n-1}$ and d_{n-1}:

$$b_0=0,\ \text{and}\ b_{n-1}+2c_{n-1}(x_n-x_{n-1})+3d_{n-1}(x_n-x_{n-1})^2=0. \qquad (5.2)$$

To solve $a_i,\ b_i,\ c_i$ and d_i from equations in (1.1)-(5.1), we first let

$$h_i=x_{i+1}-x_i,\ i=0,1,\ldots,n-1.$$

$$\vec{a} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}, \ \vec{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}, \ \vec{c} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix}, \ c_n = 0, \ \vec{d} = \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{n-1} \end{bmatrix}$$

For a free spline or natural spline :
(I) Set

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & 0 & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{bmatrix}_{(n+1)\times(n+1)}$$

$$\vec{v} = \begin{bmatrix} 0 \\ 3\left[\frac{1}{h_1}(y_2-y_1) - \frac{1}{h_0}(y_1-y_0)\right] \\ 3\left[\frac{1}{h_2}(y_3-y_2) - \frac{1}{h_1}(y_2-y_1)\right] \\ \vdots \\ 3\left[\frac{1}{h_{n-1}}(y_n-y_{n-1}) - \frac{1}{h_{n-2}}(y_{n-1}-y_{n-2})\right] \\ 0 \end{bmatrix}_{(n+1)\times 1}$$

Solve $A\vec{c} = \vec{v}$ for $\vec{c}$.
(II) Evaluate:

$$b_i = \frac{1}{h_i}(a_{i+1}-a_i) - \frac{h_i}{3}(2c_i + c_{i+1}), \ i = 0,1,\ldots,n-1.$$

$$d_i = \frac{1}{3h_i}(c_{i+1}-c_i), \ i = 0,1,\ldots,n-1.$$

For a **clamped spline**:
(I) Set

$$A = \begin{bmatrix} 2h_0 & h_0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & 0 & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & h_{n-1} & 2h_{n-1} \end{bmatrix}_{(n+1)\times(n+1)}$$

$$\vec{v} = \begin{bmatrix} 3\left[\frac{1}{h_0}(y_1 - y_0) - \alpha\right] \\ 3\left[\frac{1}{h_1}(y_2 - y_1) - \frac{1}{h_0}(y_1 - y_0)\right] \\ 3\left[\frac{1}{h_2}(y_3 - y_2) - \frac{1}{h_1}(y_2 - y_1)\right] \\ \vdots \\ 3\left[\frac{1}{h_{n-1}}(y_n - y_{n-1}) - \frac{1}{h_{n-2}}(y_{n-1} - y_{n-2})\right] \\ 3\left[\beta - \frac{1}{h_{n-1}}(y_n - y_{n-1})\right] \end{bmatrix}$$

Solve $A\vec{c} = \vec{v}$ for $\vec{c}$.
(II) Same as for the Free Spline.

Example *Let* $f(x) = \sqrt{x+1}$. *Given* $(0,1),(3,2),(8,3)$, *construct a free cubic spline and a clamped cubic spline.*

1. Free cubic spline:

(I) Set up the 3×3 matrix A and the 3×1 vector $\vec{v}$: $h_0 = 3,\ h_1 = 5$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2(3+5) & 5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 16 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 0 \\ 3\left(\frac{1}{5}(3-2) - \frac{1}{3}(2-1)\right) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2}{5} \\ 0 \end{bmatrix}$$

Solve the vector $\vec{c}$:

$$A\,\vec{c} = \vec{v},\quad \vec{c} = A^{-1}\vec{v} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 16 & 5 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\frac{2}{5} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{40} \\ 0 \end{bmatrix}$$

(II) Compute the vectors $\vec{b}$ and $\vec{d}$:

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\quad \vec{c} = \begin{bmatrix} 0 \\ -\frac{1}{40} \\ 0 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} \frac{1}{3}(2-1) - \frac{3}{3}\left(2(0) + \left(-\frac{1}{40}\right)\right) \\ \frac{1}{5}(3-2) - \frac{5}{3}\left(2\left(-\frac{1}{40}\right) + 0\right) \end{bmatrix} = \begin{bmatrix} \frac{43}{120} \\ \frac{17}{60} \end{bmatrix}$$

$$\vec{d} = \begin{bmatrix} \frac{1}{3(3)}\left(\left(-\frac{1}{40}\right) - 0\right) \\ \frac{1}{3(5)}\left(0 - \left(-\frac{1}{40}\right)\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{360} \\ \frac{1}{600} \end{bmatrix}$$

$$S(x) = \begin{cases} S_0(x) = 1 + \left(\frac{43}{120}\right)x + 0 + \left(-\frac{1}{360}\right)x^3 & \text{for } 0 \le x < 3 \\ S_1(x) = 2 + \left(\frac{17}{60}\right)(x-3) + \left(-\frac{1}{40}\right)(x-3)^2 + \left(\frac{1}{600}\right)(x-3)^3 & \text{for } 3 \le x \le 8 \end{cases}$$

2. Clamped cubic spline $f'(x) = \frac{1}{2}(x+1)^{-1/2},\ f'(0) = \frac{1}{2},\ f'(8) = \frac{1}{2}\frac{1}{\sqrt{9}} = \frac{1}{6}$

(I) Set up the 3×3 matrix A and the 3×1 vector $\vec{v}$: $h_0 = 3,\ h_1 = 5$

$$A = \begin{bmatrix} 2(3) & 3 & 0 \\ 3 & 2(3+5) & 5 \\ 0 & 5 & 2(5) \end{bmatrix} = \begin{bmatrix} 6 & 3 & 0 \\ 3 & 16 & 5 \\ 0 & 5 & 10 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 3\left(\frac{1}{3}(2-1) - \frac{1}{2}\right) \\ 3\left(\frac{1}{5}(3-2) - \frac{1}{3}(2-1)\right) \\ 3\left(\frac{1}{6} - \frac{1}{5}(3-2)\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{2}{5} \\ -\frac{1}{10} \end{bmatrix}$$

Solve the vector $\vec{c}$:

$$A\,\vec{c} = \vec{v},\quad \vec{c} = A^{-1}\vec{v} = \begin{bmatrix} 6 & 3 & 0 \\ 3 & 16 & 5 \\ 0 & 5 & 10 \end{bmatrix}^{-1} \begin{bmatrix} -\frac{1}{2} \\ -\frac{2}{5} \\ -\frac{1}{10} \end{bmatrix} = \begin{bmatrix} -\frac{19}{240} \\ -\frac{1}{120} \\ -\frac{7}{1200} \end{bmatrix}$$

(II) Compute the vectors $\vec{b}$ and $\vec{d}$: $(0,1), (3,2), (8,3)$,

$$b_i = \frac{1}{h_i}(a_{i+1} - a_i) - \frac{h_i}{3}(2c_i + c_{i+1}),\ i = 0,1,\ldots,n-1.$$

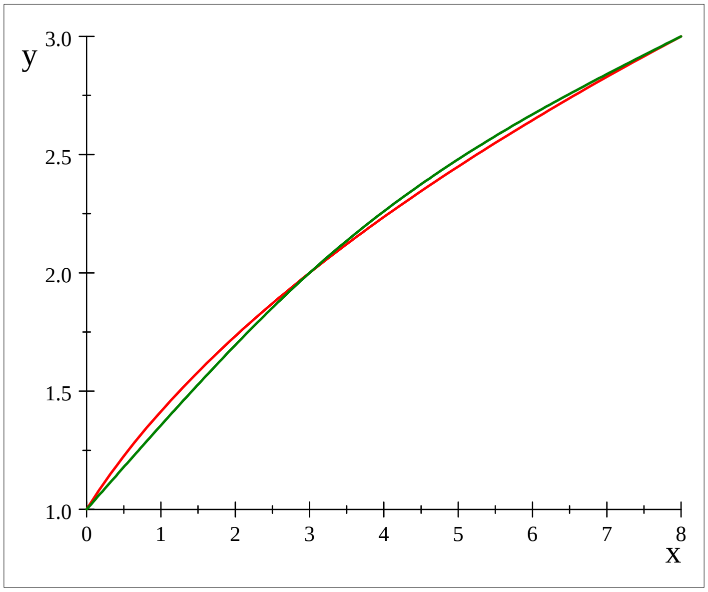
$$d_i = \frac{1}{3h_i}(c_{i+1} - c_i),\ i = 0,1,\ldots,n-1.$$

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\quad \vec{c} = \begin{bmatrix} -\frac{19}{240} \\ -\frac{1}{120} \\ -\frac{7}{1200} \end{bmatrix}$$

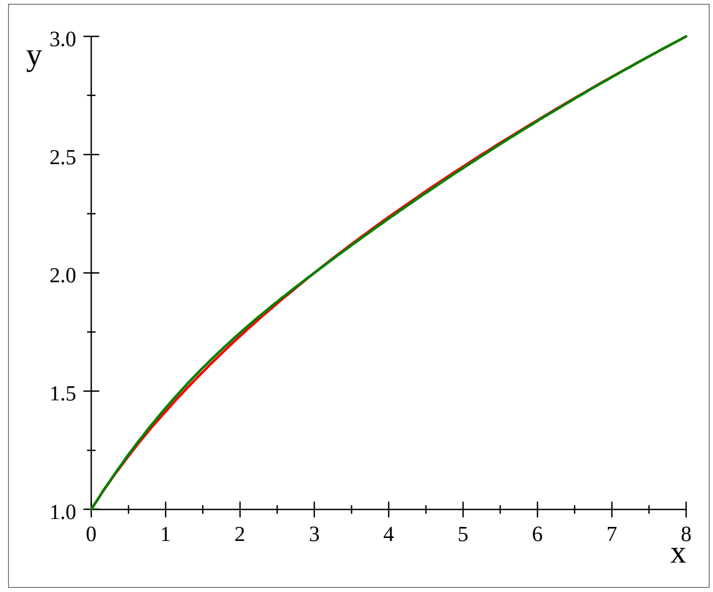
$$\vec{b} = \begin{bmatrix} \frac{1}{3}(2-1) - \frac{3}{3}\left(2\left(-\frac{19}{240}\right) + \left(-\frac{1}{120}\right)\right) \\ \frac{1}{5}(3-2) - \frac{5}{3}\left(2\left(-\frac{1}{120}\right) + -\frac{7}{1200}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{19}{80} \end{bmatrix}$$

$$\vec{d} = \begin{bmatrix} \frac{1}{3(3)}\left(\left(-\frac{1}{120}\right) - -\frac{19}{240}\right) \\ \frac{1}{3(5)}\left(-\frac{7}{1200} - \left(-\frac{1}{120}\right)\right) \end{bmatrix} = \begin{bmatrix} \frac{17}{2160} \\ \frac{1}{6000} \end{bmatrix}$$

$$S(x) = \begin{cases} S_0(x) = 1 + (\frac{1}{2})x - \frac{19}{240}x^2 + \left(\frac{17}{2160}\right)x^3 & \text{for } 0 \le x < 3 \\ S_1(x) = 2 + \left(\frac{19}{80}\right)(x-3) + \left(-\frac{1}{120}\right)(x-3)^2 + \left(\frac{1}{6000}\right)(x-3)^3 & \text{for } 3 \le x \le 8 \end{cases}$$



free cubic spline



clamped cubic spline

MatLab programs: cbspfun.m and cbspclamp.m generate a_i, b_i, c_i and d_i (av,bv,cv,dv) with input (x_i, y_i) (xv,yv). Now we use these two MatLab program for the above example.

Free cubic spline:

>> xv=[0;3;8]; yv=[1;2;3];

>> [av,bv,cv,dv]=cbspfun(xv,yv);

[A|r], where Ac=r for solving c_i:

1.000000000000000	0	0	0
3.000000000000000	16.000000000000000	5.000000000000000	-0.400000000000000
0	0	1.000000000000000	0

[av bv cv dv]

1.000000000000000	0.358333333333333	0	-0.002777777777778
2.000000000000000	0.283333333333333	-0.025000000000000	0.001666666666667

Clamped cubic spline:

>> xv=[0;3;8]; yv=[1;2;3];

>> [av,bv,cv,dv]=cbspcl(xv,yv,1/2,1/6);

Matrix A and Vector b:

6.000000000000000	3.000000000000000	0	-0.500000000000000
3.000000000000000	16.000000000000000	5.000000000000000	-0.400000000000000
0	5.000000000000000	10.000000000000000	-0.100000000000000

[av bv cv dv]

1.000000000000000	0.500000000000000	-0.079166666666667	0.007870370370370
2.000000000000000	0.237500000000000	-0.008333333333333	0.000166666666667

Example *Let* $f(x) = \cos(x^2)$, $x_0 = 0$, $x_1 = 0.6$, *and* $x_2 = 0.9$. *Find a free cubic spline and a clamped cubic spline.*

1. **Free cubic spline**:

(I) Set up the 3×3 matrix A and the 3×1 vector $\vec{v}$: $h_0 = 0.6$, $h_1 = 0.3$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0.6 & 2(0.6+0.3) & 0.3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.6 & 1.8 & 0.3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 0 \\ 3\left(\frac{1}{0.3}(\cos(0.81) - \cos(0.36)) - \frac{1}{0.6}(\cos(0.36) - 1)\right) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2.143468 \\ 0 \end{bmatrix}$$

Solve the vector $\vec{c}$:

$$A\,\vec{c} = \vec{v}, \quad \vec{c} = A^{-1}\vec{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0.6 & 1.8 & 0.3 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -2.143468 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1.19082 \\ 0 \end{bmatrix}$$

(II) Compute the vectors $\vec{b}$ and $\vec{d}$:

$$\vec{a} = \begin{bmatrix} \cos(0) \\ \cos(0.6^2) \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} 0 \\ -1.19082 \\ 0 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} \frac{1}{0.6}(\cos(0.6^2) - 1) - \frac{0.6}{3}(2(0) + (-1.19082)) \\ \frac{1}{0.3}(\cos(0.9^2) - \cos(0.6^2)) - \frac{0.3}{3}(2(-1.19082) + 0) \end{bmatrix} = \begin{bmatrix} 0.131325 \\ -0.583164 \end{bmatrix}$$

$$\vec{d} = \begin{bmatrix} \frac{1}{3(0.6)}((-1.19082) - 0) \\ \frac{1}{3(0.3)}(0 - (-1.19082)) \end{bmatrix} = \begin{bmatrix} -0.66157 \\ 1.32313 \end{bmatrix}$$

$$S(x) = \begin{cases} S_0(x) = 1 + (0.131325)x + 0 + (-0.66157)x^3 \text{ for } 0 \le x < 0.6 \\ S_1(x) = \cos(0.6^2) + (-0.583164)(x-0.6) + (-1.19082)(x-0.6)^2 + (1.32313)(x-0.6)^3 \\ \quad \text{for } 0.6 \le x \le 0.9 \end{cases}$$

2. Clamped cubic spline:

$$f'(x) = -2x\sin(x^2), \quad f'(0) = 0 = \alpha, \quad f'(0.9) = -2(0.9)\sin(0.9^2) = -1.30371 = \beta$$

(I) Set up the 3×3 matrix A and the 3×1 vector $\vec{v}$: $h_0 = 0.6$, $h_1 = 0.3$

$$A = \begin{bmatrix} 2(0.6) & 0.6 & 0 \\ 0.6 & 2(0.6+0.3) & 0.3 \\ 0 & 0.3 & 2(0.3) \end{bmatrix} = \begin{bmatrix} 1.2 & 0.6 & 0 \\ 0.6 & 1.8 & 0.3 \\ 0 & 0.3 & 0.6 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 3\left(\frac{1}{0.6}(\cos(0.6^2) - 1) - 0\right) \\ 3\left(\frac{1}{0.3}(\cos(0.9^2) - \cos(0.6^2)) - \frac{1}{0.6}(\cos(0.6^2) - \cos(0))\right) \\ 3\left(-1.30371 - \frac{1}{0.3}(\cos(0.9^2) - \cos(0.6^2))\right) \end{bmatrix} = \begin{bmatrix} -0.32052 \\ -2.1435 \\ -1.44714 \end{bmatrix}$$

Solve the vector $\vec{c}$:

$$A\,\vec{c} = \vec{v}, \quad \vec{c} = A^{-1}\vec{v} = \begin{bmatrix} 1.2 & 0.6 & 0 \\ 0.6 & 1.8 & 0.3 \\ 0 & 0.3 & 0.6 \end{bmatrix}^{-1} \begin{bmatrix} -0.32052 \\ -2.1435 \\ -1.44714 \end{bmatrix} = \begin{bmatrix} 0.19944 \\ -0.93309 \\ -1.94536 \end{bmatrix}$$

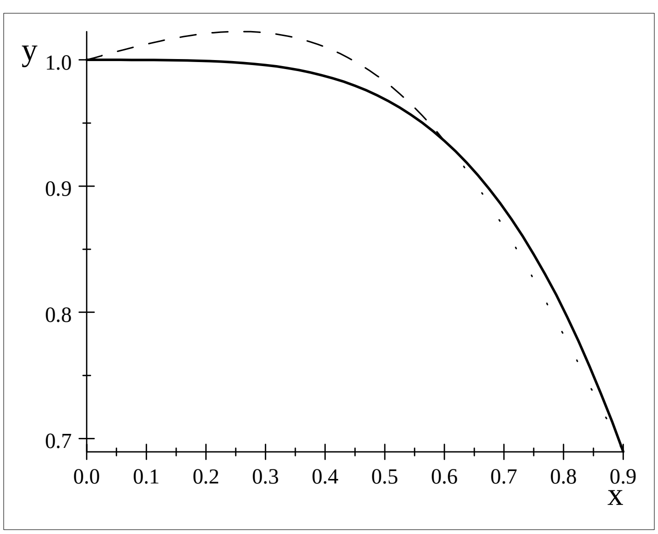
(II) Compute the vectors $\vec{b}$ and $\vec{d}$:

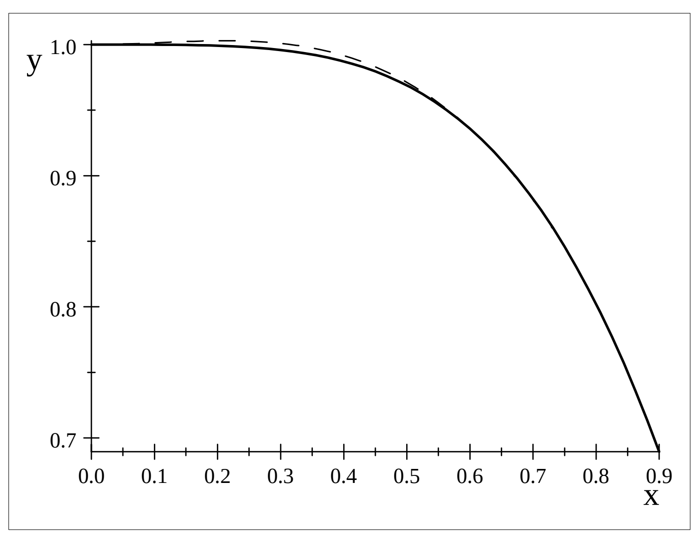
$$\vec{a} = \begin{bmatrix} \cos(0) \\ \cos(0.6^2) \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} 0.19944 \\ -0.93309 \\ -1.94536 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} \frac{1}{0.6}(\cos(0.6^2) - 1) - \frac{0.6}{3}(2(0.19944) + (-0.93309)) \\ \frac{1}{0.3}(\cos(0.9^2) - \cos(0.6^2)) - \frac{0.3}{3}(2(-0.93309) + (-1.94536)) \end{bmatrix} = \begin{bmatrix} 0.0000033728 \\ -0.44017 \end{bmatrix}$$

$$\vec{d} = \begin{bmatrix} \frac{1}{3(0.6)}((-0.93309) - 0.19944) \\ \frac{1}{3(0.3)}(-1.94536 - (-0.93309)) \end{bmatrix} = \begin{bmatrix} -0.629183 \\ -1.12474 \end{bmatrix}$$

$$S(x) = \begin{cases} S_0(x) = 1 + (0.0000033728)x + (0.19944)x^2 + (-0.629183)x^3 \text{ for } 0 \le x < 0.6 \\ S_1(x) = \cos(0.6^2) + (-0.44017)(x-0.6) + (-0.93309)(x-0.6)^2 + (-1.12474)(x-0.6)^3 \\ \quad \text{for } 0.6 \le x \le 0.9 \end{cases}$$





$S(x)$ - a free spline

$S(x)$ - a clamped spline

Use MatLab programs: cbspfun.m and cbspcl.m, for above example
Free spline:

```
>> xv=[0;0.3;0.9];
>> yv=cos(xv.^2);
>> [av,bv,cv,dv]=cbspfun(xv,yv);
```

[A|r], where Ac=r for solving c_i:

1.000000000000000	0	0	0
0.300000000000000	1.800000000000000	0.600000000000000	-1.491798830421179
0	0	1.000000000000000	0

[av bv cv dv]:

1.000000000000000	0.069386822841158	0	-0.920863475568629
0.995952733011994	-0.179246315562372	-0.828777128011766	0.460431737784314

Clamped spline:

```
>> [av,bv,cv,dv]=cbspcl(xv,yv,0,-1.30371);
```

Matrix A and Vector b

0.600000000000000	0.300000000000000	0	-0.040472669880057
0.300000000000000	1.800000000000000	0.600000000000000	-1.491798830421179
0	0.600000000000000	1.200000000000000	-2.378858499698763

[av bv cv dv]

1.000000000000000	-0.000000000000000	0.037039344878338	-0.273363260261337
0.995952733011994	-0.051584473343558	-0.208987589356866	-0.927167055026113

Applications of Spline Interpolations:

When $f(x_i) = y_i,\ i = 0, 1, \ldots, n,$ we can use $S(x)$ to approximate $f(x)$:

a. $f(x) = S(x);$

b. $f'(x) = S'(x);$

c. $\int_a^b f(x)dx = \int_a^b S(x)dx.$

Approximation error:

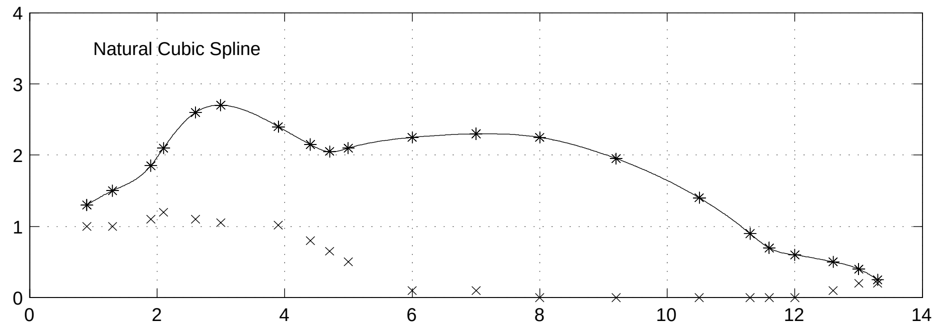
Let $f^{(4)}$ be continuous on $[a, b]$ with $\max_{a\le x\le b}|f^{(4)}(x)| = M.$ If S is the unique clamped cubic spline interpolant to f with respect to the nodes $a = x_0 < x_1 < \ldots < x_n = b,$ then

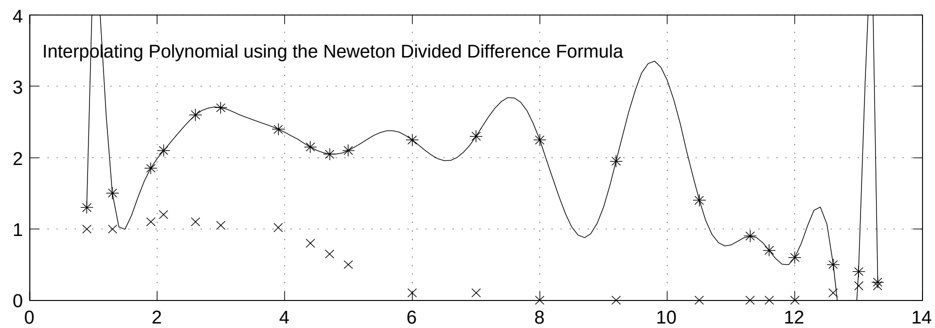
$$\max_{a\le x\le b}\left| f(x) - S(x)\right| \le \frac{5M}{384}\max_{0\le j\le n-1}(x_{j+1} - x_j)^4$$

The free spline will generally give less accurate results than the clamped conditions near the ends of the interval $\left[x_0,\ x_n\right]$ unless the function f happens to nearly satisfy $f''(x_0) = f''(x_n) = 0.$

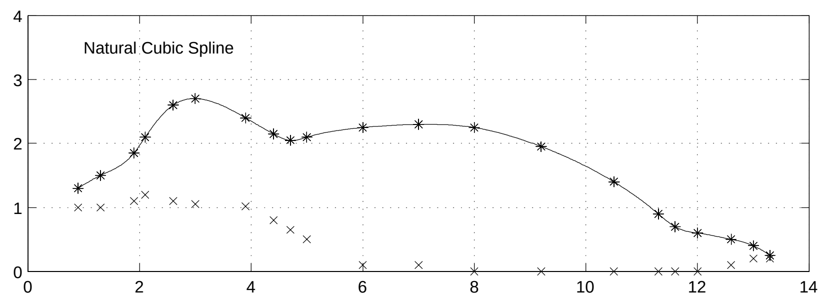
Example *Flying ruddy duck: To approximate the top profile of the duck, we have chosen points along the curve through which we want the approximating curve to pass. Notice that more points are used when the curve is changing rapidly than when it is changing slowly.*

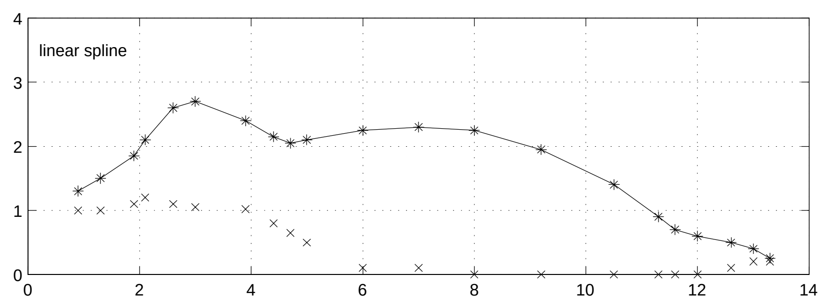
Free Spline vs Lagrangian Interpolating Polynomial:





Free Spline vs Linear Spline:





Example *Let* $S(x) = \begin{cases} S_0(x) = 1 + Bx + 2x^2 - 2x^3 & \text{if } 0 \le x < 1 \\ S_1(x) = 1 + b(x-1) - 4(x-1)^2 + 7(x-1)^3 & \text{if } 1 \le x \le 2 \end{cases}$. *Suppose that* $S(x)$ *interpolates* $f(x)$ *at* $x_0 = 0,\ x_1 = 1$ *and* $x_2 = 2$ *by a clamped cubic spline. Find* $f'(0)$ *and* $f'(2)$.

$$f'(0) = S_0'(0) = B$$
$$f'(2) = S_1'(2) = (b - 8(x-1) + 21(x-1))|_{x=2} = b - 8 + 21 = b + 13$$

Find B and b. By the conditions for a clamped cubic spline:

$$S_0(1) = S_1(1) \Rightarrow 1 + B + 2 - 2 = 1 + B = 1, \quad B = 0.$$
$$S_0'(1) = S_1'(1) \Rightarrow S_0'(1) = (B + 4x - 6x^2)|_{x=1} = 4 - 6 = -2 = S_1'(1) = b, \; b = -2$$

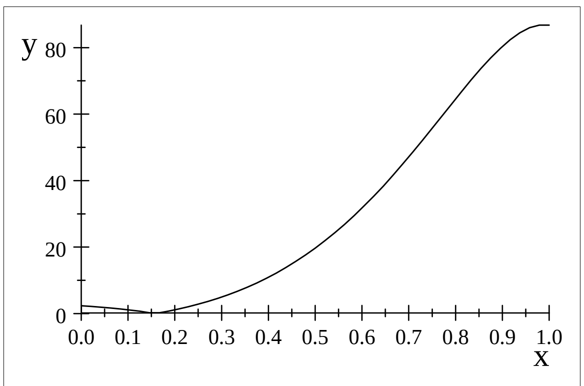
Therefore, $f'(0) = 0$ and $f'(2) = -2 + 13 = 11$.

Example *Let $f(x) = \sin(e^x - 2)$ and S be the clamped cubic spline which interpolates $f(x)$ at* $(0, -0.8415), (0.2, -0.7032), (0.5, -0.3441), (0.8, 0.2236), (1, 0.6581)$

(1) Estimate the approximation error for $f(x) \approx S(x)$ for x in $[0, 1]$.

(2) Estimate the approximation error for $\int_0^1 f(x)dx \approx \int_0^1 S(x)dx$

$$f'(x) = (\cos(e^x - 2))e^x, \; f''(x) = (-\sin(e^x - 2) + \cos(e^x - 2))e^x$$
$$f'''(x) = -\cos(e^x - 2)e^{3x} - 3\sin(e^x - 2)e^{2x} + \cos(e^x - 2)e^x$$
$$f^{(4)}(x) = (\sin(e^x - 2))e^{4x} - 6(\cos(e^x - 2))e^{3x} - 7(\sin(e^x - 2))e^{2x} + (\cos(e^x - 2))e^x$$



$$|f^{(4)}(x)| \le |f^{(4)}(1)| = 86.8 \le 87$$

$y = |f^{(4)}(x)|, \quad x$ in $[0, 1]$

$$\text{Error} = |f(x) - S(x)| \le \frac{5(87)}{384} \max_{0 \le j \le 4} \{h_j^4\} = \frac{5(87)}{384} (0.3)^4 = 0.00918$$

$$\text{Error} = \left| \int_0^1 f(x)dx - \int_0^1 S(x)dx \right| \le \int_0^1 |f(x) - S(x)| dx \le (1-0)\frac{5(87)}{384} \max_{0 \le j \le 4} \{h_j^4\} = 0.00918$$

Exercises:

1. Given the following data:

i	0	1	2	3
x_i	0	0.25	0.5	1
$y_i = f(x_i)$	1	1.4	1.6	2

a. Consider the problem of constructing a free cubic spline $S(x)$. Form the matrix A and vector $\vec{v}$ which are used to solve the vector $\vec{c}$ containing all coefficients $c_i's$. (Do not solve $\vec{c}$.)

b. Consider the problem of constructing a clamped cubic spline $S(x)$ if we know $f'(0) = 1$ and

$f'(1) = 1.2$. Form the matrix A and vector $\vec{v}$ which are used to solve the vector $\vec{c}$ containing all coefficients $c_i's$. (Do not solve $\vec{c}$.)

2. Given a cubic spline interpolation:

$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3 & \text{if } 0 \le x < 1 \\ S_1(x) = 2 + b(x-1) + c(x-1)^2 + d(x-1)^3 & \text{if } 1 \le x < 2 \end{cases},$$

determine constants $b, c,$ and d so that all conditions for a natural cubic spline hold.

3. Given a cubic spline interpolation:

$$S(x) = \begin{cases} S_0(x) = 3(x-1) + 2(x-1)^2 - (x-1)^3 & \text{if } 1 \le x < 2 \\ S_1(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3 & \text{if } 2 \le x \le 3 \end{cases}$$

and $f'(1) = f'(3)$, determine constants $a,\ b,\ c,$ and d so that all conditions for a clamped cubic spline hold.

4. Given $(x_i, y_i) = \{(0,-1),\ (1,2),\ (2,1)\}$ where $y_i = f(x_i)$ for some $f(x)$, construct a free cubic spline $S(x)$ that agrees with $f(x)$ at $x_0,\ x_1$ and x_2 (with or without using the MatLab program cbspfun.m).

a. Approximate $f(\frac{1}{2})$ by $S(\frac{1}{2})$.

b. Approximate $f'(1)$ by $S'(1)$ and $f'(\frac{3}{2})$ by $S'(\frac{3}{2})$.

c. Extra points: Approximate $\int_0^1 f(x)dx$ by $\int_0^1 S(x)dx$.

5. The clamped cubic spline $S(x)$ is constructed which agrees with the function $f(x) = x\cos(x) - 2x^2 + 3x - 1$ at data points

$$(x_i,\ y_i):\ (0.1, -0.6205),\ (0.2, -0.2840),\ (0.3, 0.0066),\ (0.4, 0.2484)$$

and satisfies conditions: $S'(0.1) = 3.585$, and $S'(0.4) = 2.1653$. Find an upper bound as small as possible for the approximation error when $f(x)$ is approximated by $S(x)$ for x in $[0.1, 0.4]$. (Don't compute $S(x)$.)

6. Suppose we are given the following data:

i	0	1	2	3	4
x_i	0	0.5	1.0	1.5	2.0
$y_i = f(x_i)$	0.5	1.425639	2.640859	4.009155	5.305472
$f'(x_i)$	1.5				2.305472

a. Use the MatLab program cbspfun.m to construct a free spline.

b. Approximate $f(0.25)$ by $S(0.25)$ and approximate $f(1.25)$ by $S(1.25)$.

c. Use the MatLab program cbspcl.m to construct a clamped spline.

d. Approximate $f(0.25)$ by $S(0.25)$ and approximate $f(1.25)$ by $S(1.25)$.

e. Suppose we know $f(x) = (x+1)^2 - 0.5e^x$. Compute the true errors to determine with spline gives a better approximation at each point.