3.3 - Divided Differences

Newton Forward Divided Differences:
Let \( P_n(x) \) be the \( n \)th Lagrange interpolating polynomial that agrees with the function \( f(x) \) at the distinct numbers \( x_0, x_1, \ldots, x_n \), i.e., \( P_n(x_i) = f(x_i) \), \( i = 0, 1, \ldots, n \). Now consider \( P_n(x) \) in the following form:

\[
P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \ldots + a_n(x-x_0) \cdots (x-x_{n-1})
\]

\[
= a_0 + \sum_{k=1}^{n} a_k \prod_{i=1}^{k} (x-x_{i-1})
\]

We need to determine \( a_0, a_1, \ldots, a_n \) such that

\[
P_n(x_i) = f(x_i), \quad i = 0, 1, \ldots, n.
\]

Observe the following:

\[
P_n(x_0) = a_0 = f(x_0)
\]

\[
P_n(x_1) = f(x_0) + a_1(x_1-x_0) = f(x_1), \quad a_1 = \frac{f(x_1)-f(x_0)}{x_1-x_0}
\]

\[
P_n(x_2) = f(x_0) + \frac{f(x_1)-f(x_0)}{x_1-x_0}(x_2-x_0) + a_2(x_2-x_0)(x_2-x_1) = f(x_2)
\]

\[
a_2 = \frac{f(x_2)-f(x_0)}{(x_2-x_0)(x_2-x_1)} - \frac{f(x_1)-f(x_0)}{(x_1-x_0)(x_2-x_0)(x_2-x_1)}(x_2-x_0)
\]

\[
= \frac{1}{x_2-x_0} \left[ \frac{f(x_2)-f(x_1)+f(x_1)-f(x_0)}{(x_2-x_1)(x_1-x_0)} - \frac{(f(x_1)-f(x_0))(x_2-x_0)}{(x_1-x_0)(x_2-x_1)} \right]
\]

Define: for \( i = 0, 1, \ldots, n \),

\[
f[x_i] = f(x_i)
\]

\[
f[x_i, x_{i+1}] = \frac{f[x_{i+1}]-f[x_i]}{x_{i+1}-x_i}
\]

\[
f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}]-f[x_i, x_{i+1}]}{x_{i+2}-x_i}
\]

\[
\vdots
\]

\[
f[x_i, x_{i+1}, \ldots, x_{i+k}] = \frac{f[x_{i+1}, \ldots, x_{i+k}]-f[x_i, \ldots, x_{i+k-1}]}{x_{i+k}-x_i}
\]

Then
\[ a_0 = f[x_0], \quad a_1 = f[x_0, x_1], \quad a_2 = f[x_0, x_1, x_2], \ldots, \quad a_n = f[x_0, x_1, \ldots, x_n] \]

and

\[ P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_{0}, x_{1}, \ldots, x_{k}] (x - x_0) \cdots (x - x_k). \]

An interpolating polynomial \( P_n(x) \) in this form is said be the **Newton’s forward divided-difference form of interpolating polynomial**. In short, we said \( P_n(x) \) is an interpolating polynomial in the **Newton’s forward divided-difference form**. The Newton’s forwarded divided-differences \( f[x_i, \ldots x_{i+k}] \) can be computed iteratively as follows.

<table>
<thead>
<tr>
<th>1st d.d.</th>
<th>2nd divided differences</th>
<th>3rd divided differences</th>
<th>4th divided diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_i )</td>
<td>( f[x_i] )</td>
<td>( f[x_i, x_{i+1}] )</td>
<td>( f[x_i, x_{i+1}, x_{i+2}] )</td>
</tr>
<tr>
<td>( x_0 )</td>
<td>( f[x_0] )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_1 )</td>
<td>( f[x_1] )</td>
<td>( f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} )</td>
<td></td>
</tr>
<tr>
<td>( x_2 )</td>
<td>( f[x_2] )</td>
<td>( f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} )</td>
<td>( f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} )</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>( f[x_3] )</td>
<td>( f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2} )</td>
<td>( f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} )</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>( f[x_4] )</td>
<td>( f[x_3, x_4] = \frac{f[x_4] - f[x_3]}{x_4 - x_3} )</td>
<td>( f[x_2, x_3, x_4] = \frac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2} )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
</tbody>
</table>

Note that the first element of each column is the coefficient of \( P_n(x) \):

\[ a_k = f[x_0, x_1, \ldots, x_k], \quad k = 0, 1, \ldots, n - 1. \]

**Example** \( f(x) = \cos(x^2), \ x_0 = 0, \ x_1 = 0.6, \ x_2 = 0.9\). Find the interpolating polynomial in the Newton forward-difference form and use it to approximate \( f(0.45) \) and \( \int_{0}^{1} f(x)dx \).

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>( f[x_i] )</th>
<th>( f[x_i, x_{i+1}] )</th>
<th>( f[x_i, x_{i+1}, x_{i+2}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_0 = 0 )</td>
<td>( 1 )</td>
<td>( \frac{\cos(0.36) - 1}{0.6 - 0} = -0.1068 )</td>
<td>( -0.1068 )</td>
</tr>
<tr>
<td>( x_1 = 0.6 )</td>
<td>( \cos(0.36) )</td>
<td>( \frac{\cos(0.36) - 1}{0.6 - 0} = -0.1068 )</td>
<td>( -0.1068(0.45) - 0.7939(0.45)(0.45 - 0.6) = 1.00552825 )</td>
</tr>
<tr>
<td>( x_2 = 0.9 )</td>
<td>( \cos(0.81) )</td>
<td>( \frac{\cos(0.81) - \cos(0.36)}{0.9 - 0.6} = -0.8213 )</td>
<td>( -0.8213 - (-0.1068) )</td>
</tr>
</tbody>
</table>

\[ P_2(x) = 1 - 0.1068x - 0.7939x(x - 0.6) \]

\[ f(0.45) \approx P_2(0.45) = 1 - 0.1068(0.45) - 0.7939(0.45)(0.45 - 0.6) = 1.00552825 \]

\[ \int_{0}^{1} P_2(x)dx = \int_{0}^{1} (1 - 0.1068(0.45) - 0.7939(0.45)(0.45 - 0.6))dx = 1.00552825 \]

Use the MatLab program **newtonfd.m** to form the table and the elements in the first row of the output.
matrix \texttt{fdmat} are coefficients \(a_0, a_1\) and \(a_2\).

\[
\begin{array}{ccc}
\gg xv=[0;0.6;0.9]; \\
\gg yv=\text{cos}(xv.^2); \\
\gg [\text{coefv,fdmat}]=\text{newtonfd}(xv,yv,3) \\
\end{array}
\]

\begin{tabular}{ccc}
coefv & 1.00000000000000 & -0.10683862720344 \\
& & -0.79387704653761 \\
fdmat & 1.00000000000000 & 0.93589682367793 \\
& & -0.10683862720344 \\
& & 0.68949843295175 \\
& & -0.82132796908729 \\
& & -0.79387704653761 \\
\end{tabular}

To evaluate \(P_2(0.45)\):

\[
\gg \text{fdmat}(1,1)+\text{fdmat}(2,2)*0.45+\text{fdmat}(3,3)*0.45*(0.45-0.6)
\]

1.00550931839974

Example For a function \(f\), the interpolating polynomial using Newton divided-difference formula is given as follows.

\[ P_2(x) = 2 + 1.5x + 1.1(x - 0.5), \]

which agrees with \(f\) at \(x_0 = 0, x_1 = 0.5\) and \(x_2 = 1\). Find \(f(1)\).

We know that \(a_0 = f[x_0] = 2, f[x_0,x_1] = 1.5\) and \(f[x_0,x_1,x_2] = 1.1\). Hence,

\begin{array}{c|c|c|c|c}
\hline
i & x_i & f[x_i] & f[x_i,x_{i+1}] & f[x_i,x_{i+1},x_{i+2}] \\
\hline
0 & 0 & 2 & & \\
1 & 0.5 & 1.5 & & \\
2 & 1 & 1.1 & & \\
\hline
\end{array}

\[ f(1) = P_2(1) = 2 + 1.5(1) + 1.1(1)(1 - 0.5) \]

\[ = 4.05 \]

Example Complete the following table.

\begin{array}{c|c|c|c}
\hline
x_i & f[x_i] & f[x_i,x_{i+1}] & f[x_i,x_{i+1},x_{i+2}] \\
\hline
x_0 = 0 & 1 & & \\
x_1 = 0.3 & f[x_1] & f[x_0,x_1] = 4 & \\
x_2 = 0.8 & f[x_2] & f[x_1,x_2] & f[x_0,x_1,x_2] = 5 \\
\hline
\end{array}

Solve \(f[x_0,x_1], f[x_1], f[x_0]\) from the following 3 equations.

\[
\begin{align*}
\frac{f[x_0,x_1]}{0.3} - \frac{f[x_0]}{0.3} &= \frac{f[x_1]}{0.3} - 1 = 4, \quad f[x_1] = 1.2 + 1 = 2.2 \\
\frac{f[x_0,x_1]}{0.8} - \frac{f[x_0,x_0]}{0.8} &= \frac{f[x_1]}{0.8} - 4 = 5, \quad f[x_1,x_2] = 4 + 4 = 8 \\
\frac{f[x_1]}{0.8} - \frac{f[x_1]}{0.3} &= \frac{f[x_2]}{0.5} - 2.2 = 8, \quad f[x_2] = 4 + 2.2 = 6.2
\end{align*}
\]
\[
\begin{array}{|c|c|c|c|}
\hline
x_i & f[x_i] & f[x_i, x_{i+1}] & f[x_i, x_{i+1}, x_{i+2}] \\
\hline
x_0 = 0 & f[x_0] = 1 & & \\
\hline
x_1 = 0.3 & f[x_1] = 2.2 & f[x_0, x_1] = 4 & \\
\hline
x_2 = 0.8 & f[x_2] = 6.2 & f[x_1, x_2] = 8 & f[x_0, x_1, x_2] = 5 \\
\hline
\end{array}
\]

**Theorem** Suppose that \( f^{(n)}(x) \) exists and continuous for \( x \) in \([a, b]\), and \( x_0, \ldots, x_n \) are distinct numbers in \([a, b]\). Then there exists a number \( c \) in \((a, b)\) such that

\[
f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(c)}{n!}.
\]

**Proof** Let \( P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, \ldots, x_{n-1}](x - x_0) \cdots (x - x_{n-1}) \) and let \( g(x) = f(x) - P_n(x) \). Then \( g(x_i) = f(x_i) - P_n(x_i) = 0 \) for \( i = 0, 1, \ldots, n \). By Rolle’s Theorem, there exists a number \( c \) in \((a, b)\) such that

\[
g^{(n)}(c) = 0.
\]

Since

\[
g^{(n)}(x) = f^{(n)}(x) - P_n^{(n)}(x) = f^{(n)}(x) - a_n n!,
\]

\[
f^{(n)}(c) - f[x_0, \ldots, x_n] n! = 0, \quad f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(c)}{n!}.
\]

**Example** Let \( f(x) = \cos(x^2) \). Find \( f[x_0, x_1, x_2] \) and \( f[x_0, x_1, x_2, x_3] \) where \( x_k = 0.1k \). Give an upper bound for each of \( |f[x_0, x_1, x_2]| \) and \( |f[x_0, x_1, x_2, x_3]| \).

\[
x_0 = 0, \ x_1 = 0.1, \ x_2 = 0.2 \text{ and } x_3 = 0.3
\]

\[
f[x_0, x_1, x_2] = \frac{f''(c)}{2!} \text{ where } c \text{ is in } [0, 0.2] \text{ and } f[x_0, x_1, x_2, x_3] = \frac{f'''(c)}{3!} \text{ where } c \text{ is in } [0, 0.3].
\]

\[
f'(x) = -2x \sin(x^2), \quad f''(x) = -4x^2 \cos(x^2) - 2 \sin(x^2), \quad f'''(x) = 8x^3 \sin(x^2) - 12x \cos(x^2)
\]

\[
f[x_0, x_1, x_2] = \frac{-4c^2 \cos(c^2) - 2 \sin(c^2)}{2} \text{ where } c \text{ is in } [0, 0.2].
\]

\[
f[x_0, x_1, x_2, x_3] = \frac{8c^3 \sin(c^2) - 12x \cos(c^2)}{6} \text{ where } c \text{ is in } [0, 0.3].
\]

\[
|f[x_0, x_1, x_2]| \leq \frac{1}{2} \left(4(0.2)^2(1) + 2(1)\right) = 1.08
\]

\[
|f[x_0, x_1, x_2, x_3]| \leq \frac{1}{6} \left(8(0.3)^3(1) + 12(0.3)(1)\right) = 0.636
\]
Exercises:

1. Given \((x_i, y_i) = \{(0, -1), \left(\frac{1}{2}, 1\right), (1, 2)\}\) where \(y_i = f(x_i)\) for some \(f(x)\), construct the interpolating polynomial \(P_2(x)\) that agrees with \(f(x)\) at \(x_0, x_1\) and \(x_2\) in the Newton’s forward divided-difference form.

2. For a function \(f\), the forward divided differences are given in the following table. Complete the table.

<table>
<thead>
<tr>
<th>(x_i)</th>
<th>(f(x_i))</th>
<th>(f[ x_i, x_{i+1}])</th>
<th>(f[ x_i, x_{i+1}, x_{i+2}])</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>6</td>
<td>10</td>
<td>(\frac{50}{7})</td>
</tr>
</tbody>
</table>

3. Suppose we know that \(f(0) = 1, f(0.25) = 1.64872, f(0.5) = 2.71828, f(0.75) = 4.48169\). Use given data to approximate \(f(0.43)\) by an interpolating polynomial in the Newton’s forward divided-difference form. (MatLab program newtonfd.m).

4. Let \(f(x) = \sqrt{x + 1}\) and \(x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75\). Find \(f[ x_0, x_1, x_2]\) and \(f[ x_0, x_1, x_2, x_3]\). Give an upper bound for each of \(|f[ x_0, x_1, x_2]|\) and \(|f[ x_0, x_1, x_2, x_3]|\).

5. Extra points. Show that \(f[ x_0, x_1, \ldots, x_n, x] = \frac{f^{(n+1)}(c(x))}{(n + 1)!}\) for some \(c(x)\) in \([x_0, x_n]\). (Hint: From Section 3.1,

\[
f(x) = P_n(x) + \frac{f^{(n+1)}(c(x))}{(n + 1)!} (x - x_0)(x - x_1)\cdots(x - x_n).
\]

Now consider the interpolating polynomial of degree \(n + 1\) with \(x_0, \ldots, x_n, x\).)