3.4 - Piecewise Linear-Quadratic Interpolation

Piecewise-polynomial Approximation:

**Problem:** Given \( n + 1 \) pairs of data points \( (x_i, y_i), \ i = 0, 1, \ldots, n, \) find a piecewise-polynomial \( S(x) \)

\[
S(x) = \begin{cases} 
S_0(x) & \text{if } x_0 \leq x \leq x_1 \\
S_1(x) & \text{if } x_1 \leq x \leq x_2 \\
& \vdots \\
S_{n-1}(x) & \text{if } x_{n-1} \leq x \leq x_n 
\end{cases}
\]

where \( S_i(x) \) are polynomials and \( S(x_i) = y_i, \ i = 0, 1, \ldots, n. \)

Observe that each \( S_i \) can be in a different degree of polynomial or all \( S_i \) are in a same degree.

1. **Linear Splines:**

\[
L_i(x) = a_i + b_i(x - x_i), \ i = 0, 1, \ldots, n - 1, \quad \text{and } L(x) = \begin{cases} 
L_0(x) & \text{if } x_0 \leq x \leq x_1 \\
& \vdots \\
L_{n-1}(x) & \text{if } x_{n-1} \leq x \leq x_n 
\end{cases}
\]

2. **Quadratic Splines:**

\[
Q_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2, \ i = 0, 1, \ldots, n - 1, \quad \text{and } Q(x) = \begin{cases} 
Q_0(x) & \text{if } x_0 \leq x \leq x_1 \\
& \vdots \\
Q_{n-1}(x) & \text{if } x_{n-1} \leq x \leq x_n 
\end{cases}
\]

3. **Cubic Splines:** (Section 3.4)

\[
S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3, \ i = 0, 1, \ldots, n - 1.
\]

**Question:** How can we solve coefficients \( a_i, \) and \( b_i \) for a linear spline; \( a_i, b_i, \) and \( c_i \) for a quadratic spline; and \( a_i, b_i, c_i, \) and \( d_i \) for a cubic spline?

1. **Linear Splines:**

\[
L_i(x) = a_i + b_i(x - x_i), \ i = 0, 1, \ldots, n - 1.
\]

We need to find \( 2n \) unknowns: \( a_0, a_1, \ldots, a_{n-1}, \ b_0, b_1, \ldots, b_{n-1} \) so that \( L_i(x) \) satisfy the following conditions:

1. \( L_i(x_i) = y_i, \ i = 0, 1, 2, \ldots, n - \) condition for interpolation, a total of \( n + 1 \) equations
2. \( L_i(x_{i+1}) = L_{i+1}(x_{i+1}), \ i = 0, 1, \ldots, n - 2 - \) condition for continuity at interior points, a total of \( n - 1 \) equations

Since we have \( 2n \) equations for \( 2n \) unknowns, we can solve \( a_i \) and \( b_i \) uniquely. Observe the following. From the condition in (1), \( L_i(x_i) = y_i, i = 0, 1, 2, \ldots, n, \) we can obtain \( a_i \) for \( i = 0, 1, \ldots, n - 1: \)

\[
L_i(x_i) = a_i = y_i, \ i = 0, 1, 2, \ldots, n - 1
\]

and \( b_{n-1} \)

\[
L_{n-1}(x_n) = b_{n-1}(x_n - x_{n-1}) + y_{n-1} = y_n, \quad b_{n-1} = \frac{y_n - y_{n-1}}{x_n - x_{n-1}}.
\]

From the the condition in (2), \( L_i(x_{i+1}) = L_{i+1}(x_{i+1}), \ i = 0, 1, \ldots, n - 2, \) we can solve \( b_i \) for \( i = 0, 1, \ldots, n - 2: \)

\[
L_i(x_{i+1}) = b_i(x_{i+1} - x_i) + y_i = L_{i+1}(x_{i+1}) = y_{i+1}, \quad b_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}, \ i = 0, 1, \ldots, n - 2.
\]
Theorem Let \( a = x_0 < x_1 < \cdots < x_n = b, \ h = \max_{0 \leq k \leq n-1} |x_{k+1} - x_k| \) and \( f''(x) \) be continuous on \([a,b]\). Then

\[
\max_{x \in [a,b]} |f(x) - L(x)| \leq \frac{1}{8} h^2 \max_{c \in [a,b]} |f''(c)|.
\]

Proof Let \( x \in [x_i, x_{i+1}] \). Then \( f(x) \approx L_i(x) = y_i + \frac{y_{i+1} - y_i}{x_{i+1} - x_i} (x - x_i) \). Note that the Newton Form

\[
f(x) = f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} (x - x_i) + \frac{f''(c_i(x))}{2!} (x - x_i)(x - x_{i+1}).
\]

Note that \( y_i = f(x_i), \ i = 0, \ldots, n \). So,

\[
f(x) = L_i(x) + \frac{f''(c_i(x))}{2!} (x - x_i)(x - x_{i+1}).
\]

Approximation error:

\[
|f(x) - L(x)| = |f(x) - L_i(x)| = \left| \frac{f''(c_i(x))}{2!} (x - x_i)(x - x_{i+1}) \right|
\]

\[
\leq \frac{1}{2} \max_{c_i \in [x_i, x_{i+1}]} \left| f''(c_i(x)) \right| \max_{x \in [x_i, x_{i+1}]} |(x - x_i)(x - x_{i+1})|.
\]

What is \( \max_{x \in [x_i, x_{i+1}]} |(x - x_i)(x - x_{i+1})| \)? Let \( q_i(x) = (x - x_i)(x - x_{i+1}) \). The graph of \( q_i(x) \) is a concave up parabola so the vertex \((\bar{x}_i, \bar{y}_i)\) of the parabola is the lowest point on the graph. Hence, \( \max_{x \in [x_i, x_{i+1}]} |(x - x_i)(x - x_{i+1})| = |\bar{y}_i| \). Because of the symmetry of a parabola,

\[
x_i = \frac{1}{2} (x_i + x_{i+1})
\]

and then

\[
|\bar{y}_i| = \left( \frac{1}{2} (x_i + x_{i+1}) - x_i \right) \left( x_{i+1} - \frac{1}{2} (x_i + x_{i+1}) \right) = \frac{1}{4} (x_{i+1} - x_i)^2 \leq \frac{1}{4} h^2.
\]

Hence,

\[
|f(x) - L(x)| \leq \frac{1}{2} \max_{c_i \in [x_i, x_{i+1}]} \left| f''(c_i) \right| \max_{x \in [x_i, x_{i+1}]} |(x - x_i)(x - x_{i+1})|
\]

\[
\leq \frac{1}{2} \max_{c \in [x_i, x_{i+1}]} \left| f''(c) \right| \left( \frac{1}{4} h^2 \right)
\]

\[
\leq \frac{1}{8} h^2 \max_{c \in [a,b]} \left| f''(c) \right|
\]

Note:

If we know \( x \in [x_i, x_{i+1}] \), then

\[
|f(x) - L(x)| = |f(x) - L_i(x)| \leq \frac{1}{8} (x_{i+1} - x_i)^2 \max_{x \in [x_i, x_{i+1}]} |f''(x)|.
\]
Example Let \( f(x) = \sqrt{x} + 1 \). Given \((0, 1), (3, 2), (8, 3)\),

(i) construct a linear spline \( L(x) \);

(ii) approximate \( f(2) \) by \( L(2) \) and \( f(5) \) by \( L(5) \); and

(iii) estimate the approximate errors for \( |f(2) - L(2)|, |f(5) - L(5)| \) and \( |f(x) - L(x)| \).

(i) \( L(x) = \begin{cases} 
L_0(x) = a_0 + b_0x & \text{if } 0 \leq x \leq 3 \\
L_1(x) = a_1 + b_1(x - 3) & \text{if } 3 \leq x \leq 8 
\end{cases} \)

\( \begin{align*}
L_0(x) &= 1 + \frac{1}{3}x \\
L_1(x) &= 2 + \frac{1}{5}(x - 3)
\end{align*} \)

(ii) \( f(2) \approx L_0(2) = 1 + \frac{2}{3} = \frac{5}{3} \)

\( f(5) \approx L_1(5) = 2 + \frac{1}{5}(5 - 3) = \frac{12}{5} \)

(iii) \( f'(x) = \frac{1}{2}(x + 1)^{-1/2}, f''(x) = -\frac{1}{4}(x + 1)^{-3/2}, |f''(x)| = \left| -\frac{1}{4}(x + 1)^{-3/2} \right| = \frac{1}{4} \sqrt{\frac{1}{(x + 1)^3}}. \)

For \( x = 2 \),

\( |f(2) - L(2)| \leq \frac{1}{8} (3 - 0)^2 \max_{c \in [0,3]} \left| \frac{1}{4} \sqrt{\frac{1}{(x + 1)^3}} \right| \leq \frac{1}{8} \sqrt{\frac{1}{3^3}} \left( \frac{1}{4} \right) = 0.28125, \)

and for \( x = 5 \),

\( |f(5) - L(5)| \leq \frac{1}{8} (8 - 3)^2 \max_{c \in [3,8]} \left| \frac{1}{4} \sqrt{\frac{1}{(x + 1)^3}} \right| \leq \frac{1}{8} \sqrt{\frac{1}{5^3}} \left( \frac{1}{4} \right) = 0.097656. \)

To see how accurate these approximations are, we check the true errors:

\( |f(2) - L(2)| = \left| \sqrt{2} + 1 - \frac{5}{3} \right| = 0.06538 \)

\( |f(5) - L(5)| = \left| \sqrt{5} + 1 - \frac{12}{5} \right| = 0.04949. \)

Now for any \( x \) in \([0,8]\),

\( |f(x) - L(x)| \leq \frac{1}{8} h^2 \max_{c \in [0,8]} \left| f''(c) \right| \)

\( \leq \frac{1}{8} \left( \frac{1}{4} \right)^2 \sqrt{\frac{1}{(x + 1)^3}} \left( \frac{1}{4} \right) = 0.78125. \)
2. Quadratic Splines:

\[ Q_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2, \quad i = 0, 1, \ldots, n-1. \]

We need to find 3n unknowns: \( a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}, c_0, c_1, \ldots, c_{n-1} \) so that \( Q_i(x) \) satisfy the following conditions:

1. \( Q_i(x_i) = y_i, \quad i = 0, 1, 2, \ldots, n \) - condition for interpolation, a total of \( n + 1 \) equations
2. \( Q_i(x_{i+1}) = Q_{i+1}(x_{i+1}), \quad i = 0, 1, \ldots, n-2 \) - condition for continuity at interior points, a total of \( n - 1 \) equations
3. \( Q_i'(x_{i+1}) = Q_{i+1}'(x_{i+1}), \quad i = 0, 1, \ldots, n-2 \) - condition for continuous slope at interior points - \( n - 1 \) equations
4. one additional condition from the following three conditions:
   - (4.1) \( Q'(x_0) = \alpha \)
   - (4.2) \( Q'(x_n) = \beta \)
   - (4.3) \( Q'(x_0) = Q'(x_n) \)
   depending on information about \( f(x) \), 1 equation

Since we have 3n equations for 3n unknowns, we can again solve \( a_i, b_i \) and \( c_i \) uniquely. From the condition in (1), \( Q_i(x_i) = y_i, i = 0, 1, 2, \ldots, n \), we can obtain \( a_i \) for \( i = 0, 1, \ldots, n-1 \):

\[ Q_i(x_i) = a_i = y_i, \quad i = 0, 1, 2, \ldots, n-1, \]

and an equation in \( b_{n-1} \) and \( c_{n-1} \):

\[ Q_{n-1}(x_{n-1}) = y_{n-1} + b_{n-1}(x_{n-1} - x_{n-1}) + c_{n-1}(x_{n-1} - x_{n-1})^2 = y_n. \]  
(1.1)

From the condition in (2), \( Q_i(x_{i+1}) = Q_{i+1}(x_{i+1}), \quad i = 0, 1, \ldots, n-2 \), we can obtain \( n - 1 \) equations in \( b_i \) and \( c_i \) for \( i = 0, 1, \ldots, n-2 \):

\[ Q_i(x_{i+1}) = y_i + b_i(x_{i+1} - x_i) + c_i(x_{i+1} - x_i)^2 = Q_{i+1}(x_{i+1}) = y_{i+1}, \quad i = 0, 1, \ldots, n-2. \]  
(2.1)

Similarly, from the condition in (3), \( Q_i'(x_{i+1}) = Q_{i+1}'(x_{i+1}), \quad i = 0, 1, \ldots, n-2 \), we can obtain \( n - 2 \) equations in \( b_i \) and \( c_i \) for \( i = 0, 1, \ldots, n-2 \):

\[ Q_i'(x_{i+1}) = b_i + 2c_i(x_{i+1} - x_i) = Q_{i+1}'(x_{i+1}) = b_{i+1}, \quad i = 0, 1, \ldots, n-2. \]  
(3.1)

with one more equation from the condition given in (4),

\[ Q'(x_0) = Q_0'(x_0) = b_0 + 2c_0(x_1 - x_0) = \alpha \]

\[ Q'(x_n) = Q_n'(x_n) = b_{n-1} + 2c_{n-1}(x_n - x_{n-1}) = \beta \]

or

\[ Q'(x_0) = b_0 + 2c_0(x_1 - x_0) = Q'(x_n) = b_{n-1} + 2c_{n-1}(x_n - x_{n-1}), \]

we have 2n equations to solve \( b_i \)'s and \( c_i \)'s.

**Example** Let \( f(x) = \sqrt{x+1} \). Given \((0,1), (3,2), (8,3)\)

(i) construct a quadratic spline \( Q(x) \) suppose that we also know \( f'(0) = \frac{1}{2} \);

(ii) approximate \( f(2) \) by \( Q(2) \) and \( f(5) \) by \( Q(5) \);

(iii) approximate \( f'(3) \) by \( Q'(3) \) and \( f'(5) \) by \( Q'(5) \); and

(iv) approximate \( \int_0^3 f(x)dx \) by \( \int_0^3 Q(x)dx \).

(i)
We solve $b_0$, $b_1$, $c_0$ and $c_1$ as follows. From the condition (1.1):

$$Q_1(8) = 2 + b_1(5) + c_1(5)^2 = 2 + 5b_1 + 25c_1 = y_2 = 3.$$ (1.2)

From the condition in (2.1), we have the equation:

$$Q_0(3) = 1 + b_0(3) + c_0(3)^2 = 1 + 3b_0 + 9c_0 = Q_1(3) = y_1 = 2.$$ (2.2)

From the condition (3.1), we have the equation:

$$Q_0'(3) = b_0 + 2c_0(3) = b_0 + 6c_0 = Q_1'(3) = b_1.$$ (3.2)

From the condition (4.1), we have the equation:

$$Q_0'(0) = b_0 = \frac{1}{2}.$$ (4.1)

Solve $c_0$ in (2.2):

$$c_0 = \frac{1}{9}(2 - 1 - 3b_0) = \frac{1}{9} \left(1 - \frac{3}{2}\right) = -\frac{1}{18}.$$ (2.2)

Solve $b_1$ in (3.2):

$$b_1 = \frac{1}{2} + 6 \left(-\frac{1}{18}\right) = \frac{1}{6}.$$ (3.2)

and solve $c_1$ in (1.2):

$$c_1 = \frac{1}{25} \left(3 - 2 - \frac{5}{6}\right) = \frac{1}{150}.$$ (1.2)

Hence,

$$Q(x) = \begin{cases} Q_0(x) = 1 + \frac{1}{2}x - \frac{1}{18}x^2 & \text{if } 0 \leq x \leq 3 \\ Q_1(x) = 2 + \frac{1}{6}(x - 3) + \frac{1}{150}(x - 3)^2 & \text{if } 3 \leq x \leq 8 \end{cases}$$

(ii) $f(2) \approx Q(2) = Q_0(2) = 1 + \frac{1}{2}(2) - \frac{1}{18}(2^2) = \frac{16}{9} = 1.77777778$

$f(5) \approx Q(5) = Q_1(5) = 2 + \frac{1}{6}(2) + \frac{1}{150}(2)^2 = \frac{59}{25} = 2.36$

True errors:

$$\left|\sqrt{2 + 1} - \frac{16}{9}\right| = 0.04573, \quad \left|\sqrt{5 + 1} - \frac{59}{25}\right| = 0.0894897.$$
(iii) \( f'(3) \approx Q'(3) = Q'_0(3) = Q'_1(3) = \frac{1}{6} \)
\( f'(5) \approx Q'(5) = Q'_1(5) = \frac{1}{6} + \frac{2}{150} (5 - 3) = \frac{29}{150} = 0.193333. \)
(iv) \( \int_0^3 f(x) \, dx \approx \int_0^3 Q(x) \, dx = \int_0^3 Q_0(x) \, dx = \int_0^3 \left( 1 + \frac{1}{2} x - \frac{1}{18} x^2 \right) \, dx = 4.75 \)

**Exercises:**

1. The following table gives the viscosity of sulfuric acid, in millipascal-seconds (centipoises), as a function of concentration, in mass percent:

<table>
<thead>
<tr>
<th>Concentration (C)</th>
<th>0</th>
<th>20</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Viscosity (V)</td>
<td>0.80</td>
<td>1.40</td>
<td>2.51</td>
<td>5.37</td>
<td>17.4</td>
<td>24.2</td>
</tr>
</tbody>
</table>

   a. Using given date, construct a linear spline \( L(C) \) to approximate the viscosity \( V(C) \).

   b. Estimate the viscosity by \( L(C) \) when the concentration is 5% and 63% and 92%.

2. Let \( f(x) = e^x \). Given \((-1, e^{-1}), (0, 1), (1, e)\), construct a linear spline \( L(x) \) to approximate \( f(x) \).

   a. Approximate \( f(-\frac{1}{2}) \) by \( L(-\frac{1}{2}) \) and \( f(\frac{1}{2}) \) by \( L(\frac{1}{2}) \); and

   b. Estimate the approximate errors for \( |f(-\frac{1}{2}) - L(-\frac{1}{2})|, |f(\frac{1}{2}) - L(\frac{1}{2})| \) and \( |f(x) - L(x)| \).

3. Given \((x_i, y_i) = \{(0, -1), (1, 2), (2, 1)\}\) where \( y_i = f(x_i) \) for some \( f(x) \), construct a quadratic spline \( Q(x) \) to approximate \( f(x) \) if we also know \( f''(0) = 3 \).

   a. Approximate \( f(\frac{1}{2}) \) by \( Q(\frac{1}{2}) \) and \( f(\frac{3}{2}) \) by \( Q(\frac{3}{2}) \).

   b. Approximate \( f'(1) \) by \( Q'(1) \) and \( f''(\frac{3}{2}) \) by \( Q'(\frac{3}{2}) \).

   c. Approximate \( \int_0^1 f(x) \, dx \) by \( \int_0^1 Q(x) \, dx \).