Exercises:
1. Let \( g(x) = x - \frac{x^3 - 5}{3x^2} \) and \( p_0 = 1 \). Compute (without using the MatLab program fixpt.m) \( p_1 \) and \( p_2 \).

\[
p_1 = g(1) = 1 - \frac{1^3 - 5}{3(1)^2} = \frac{7}{3}
\]

\[
p_2 = g\left(\frac{7}{3}\right) = 1 - \frac{\left(\frac{7}{3}\right)^3 - 5}{3\left(\frac{7}{3}\right)^2} = \frac{233}{441} = 0.528344671
\]

2. The graph of \( g \) for \( x \) in \([0, 1]\) is given at the left. Let \( p_0 = 0 \). Compute graphically \( p_1, p_2, \) \( p_3 \) generated by the Fixed-point Algorithm.

3. Consider solving the equation \( x^3 - x - 1 = 0 \) for \( x \) in \([1, 2]\).

   (1) Derive a function \( g(x) \) such that the solution of the equation is a fixed-point of \( g(x) \). Show your derivation.

   (2) Determine by the Fixed-Point Theorem if the sequence of the iterations generated by your \( g(x) \) converges to the solution of the equation. Find a different \( g(x) \) if it does not converge.

   (3) Solve the equation using the Fixed-point Algorithm (fixed.m) with your \( g(x) \) within \( 10^{-8} \) with \( p_0 = 1 \).

   (1) \( x^3 - x - 1 = 0 \), \( x(x^2 - 1) = 1 \), \( x = \frac{1}{x^2 - 1} \), \( g_1(x) = \frac{1}{x^2 - 1} \).

   (2) \( g_1(x) \) is not defined at \( x = 1 \) and \( \lim_{x \to 1} g_1(x) = \infty \) so we cannot determine it converges by the Fixed-Point Theorem. Define another \( g(x) \):

   \[
x^3 - x - 1 = 0, \quad x^3 = 1 + x, \quad x = \sqrt[3]{1 + x}, \quad g_2(x) = \sqrt[3]{1 + x}.
\]

   \[
g_2(1) = \sqrt[3]{2} = 1.25992105 > 1 \) and \( g_2(2) = \sqrt[3]{3} = 1.44224957 < 2 \) so \( 1 \leq g_1(x) \leq 2 \)

   \[
g_2(x) = \frac{1}{3\sqrt[3]{(1 + x)^2}} \text{ and } |g_2'(x)| \leq \frac{1}{3\sqrt[3]{(1 + 1)^2}} = 0.209986842 < 1
\]
By the Fixed-Point Theorem, \( \{p_n\} \) generated by \( g_2 \) converges to its fixed point.

(3) Numerical results:

```
>> gfun=@(x) (x+1).^(1/3);
>> fixpt
input initial point \( p_0 = 1 \)
input the tolerance for stopping criterion, ex:.0001,10^(-8) 10^(-5)
input the maximum number of iterations 100
Algorithm converges with number of iterations = 8
fixed point \( p = 1.324717372435671 \)
```

4. Consider solving the equation \( \cos(x) - x = 0 \) for \( x \) in \( [0, \frac{\pi}{2}] \).

a. Show that the equation \( \cos(x) - x = 0 \) has a unique solution in \( [0, \frac{\pi}{2}] \) by two steps:
   (i) show the equation has a solution in \( [0, \frac{\pi}{2}] \) by the Intermediate Value Theorem;
      Let \( f(x) = \cos(x) - x \). \( f(0) = 1.0 > 0 \) and \( f\left(\frac{\pi}{2}\right) = -1.57079633 < 0 \).
      Since \( f(0)f\left(\frac{\pi}{2}\right) < 0 \), by the IVT, there is a solution in \( [0, \frac{\pi}{2}] \).
   (ii) show the solution is unique by Rolle’s Theorem.
      \( f'(x) = -\sin(x) - 1 < 0 \) for \( x \) in \( [0, \frac{\pi}{2}] \), the solution is unique.

b. Approximate the solution of the equation within \( 10^{-8} \) by the following methods:
   Report the number of iterations for each method. Rank the methods based on numbers of iterations.
   (i) the Bisection Method (bisect.m);
      ```
      >> fun=@(x) cos(x)-x
      >> bisect
      the left end point \( a = 0 \)
      the right end point \( b = \pi/2 \)
      epsilon = 10^(-8)
      the approximation to the solution -
      xN = 0.739085132102370
      function value at the solution -
      fxN = 1.862379916950374e-009
      number of iterations - \( N_{it} = 28 \)
      (ii) the Newton Method (newton.m) with \( p_0 = 0 \);
      ```
      ```
      >> fpfun=@(x) -sin(x)-1;
      >> [xv,flag,ct]=newton(fun,fpfun,0,10^(-8),100)
      xv =
      0
      1.000000000000000
      0.750363867840244
      0.739112890911362
      0.739085133385284
      flag = 1
      ct = 5
      ```
(iii) the Fixed Point Method (fixpt.m) with \( g(x) = \cos(x) \) and \( p_0 = 0 \).

\[
\gg \text{gfun}=@(x) \cos(x);
\]

\[
\gg \text{fixpt}
\]

input initial point \( p_0 = 0 \)

input the tolerance for stopping criterion, ex: \( .0001, 10^{-8}, 10^{-8} \)

input the maximum number of iterations 100

Algorithm converges with number of iterations = 47

fixed point \( p = 0.739085136646572 \)

Based on the numbers of iterations, (1) Newton Method (2) Bisection Method (3) Fixed-Point Algorithm

\[c. \ \text{Estimate (the best you can) the asymptotic error constant } \lambda \text{ for the Newton Method. Does } \lambda \text{ match with the performance of the Newton Method?}\]

\[\begin{align*}
\text{With } \alpha = 1, \lambda = 0, \text{ with } \alpha = 2, \lambda = 0.225.
\end{align*}\]

5. Consider the function \( g(x) = 1 + x - \frac{1}{8}x^3 \).

\(a. \) Show that \( g(x) \) has a unique fixed point on the real line.

Graphically, we show that \( g(x) \) has a unique fixed point on the real line.

\[
\begin{align*}
g(x) &= 1 + x - \frac{1}{8}x^3 \\
g(x) &= 1 + x - \frac{1}{8}x^3, \quad \left[ -\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}} \right]
\end{align*}\]

\(b. \) Can we show \( g(x) \) has a unique fixed point using the theorem for the existence and uniqueness?
Explain.

Yes. $g'(x) = 1 - \frac{3}{8}x^2$. $|g'(x)| = |1 - \frac{3}{8}x^2| < 1$ if and only if

$$-1 < 1 - \frac{3}{8}x^2 < 1 \iff -2 < -\frac{3}{8}x^2 < 0 \text{ or } 0 < \frac{3}{8}x^2 < 2 \text{ or } 0 < x^2 < \frac{16}{3} \text{ or } |x| < \frac{4}{\sqrt{3}} = 2.309$$

Observe (graphically) that in the interval $\left[-\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right]$, $-\frac{4}{\sqrt{3}} < g(x) < \frac{4}{\sqrt{3}}$.

Since also $|g'(x)| < 1$ for $x$ in $\left[-\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right]$, $g(x)$ has a unique fixed-point in $\left[-\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right]$. 