1.3 - Algorithms and their Convergence

1. Algorithms

What is an algorithm?

Definition An algorithm is a precisely defined sequence of steps for performing a specified task.

What are algorithms for? What are our tasks that need algorithms? In Lectures (1.2), we use algorithms to approximate \( \pi \) and to calculate sines and cosines. There are many tasks that need to be accomplished by algorithms. For this course, we study algorithms for the following tasks.

a. Solve equations: \( f(x) = 0 \).

b. Find a polynomial \( P(x) \) that is closest to \( f(x) \) with a set of values of \( f(x) \) for \( x \) in \([a, b]\).

c. Find a best approximation of \( f'(a) \) using a set of \( f(x) \) where \( x \) is near \( a \).

d. Approximate \( \int_a^b f(x) \, dx \) when the antiderivative of \( f(x) \) is not known.

Example Find the real solution of \( x^3 + 2x^2 + 10x = 20 \).

The solution is called "Fibonacci’s forgotten number".

Example Solve \( \frac{1}{\sqrt{x}} = \ln \left( \sqrt{\frac{1}{1-x}} + \sqrt{\frac{x}{1-x}} \right) \) for \( x \).

This problem is given by Dr. Groetsch in one of his journal article for finding the maximal path length.

Let \( x^* \) be the solution of a problem we are looking for. An algorithm generates a sequence of approximations \( \{x_n\} \) to \( x^* \) iteratively.

2. Convergence of an Algorithm

For a given problem, we like to have an algorithm which solves the problem efficiently. How can we evaluate the efficiency of an algorithm? Let the sequence of approximation \( \{x_n\} \) to the solution \( x^* \) be generated by an algorithm. Assume that the sequence converges to the solution \( x^* \). An algorithm is efficient if \( |x_n - x^*| \to 0 \) fast (for example, exponentially vs. linearly). There are two ways to measure the efficiency of an algorithm:

a. rate of convergence; and

b. order of convergence.

a. Rate of Convergence:

Definition Let \( \{x_n\}_{n=1}^\infty \) converge to a number \( x^* \). Suppose that \( \{\beta_n\}_{n=1}^\infty \) is a sequence known to converge to 0. The sequence \( \{x_n\}_{n=1}^\infty \) is said to converge to \( x^* \) with rate of convergence \( O(\beta_n) \) or \( x_n = x^* + O(\beta_n) \) if there exists a positive constant \( K \) such that

\[
\left| x_n - x^* \right| \leq K \left| \beta_n \right| \text{ for all sufficiently large } n.
\]

Note that the above inequality implies that

\[
\lim_{n \to \infty} \frac{|x_n - x^*|}{|\beta_n|} = K.
\]

Note that usually we choose \( \beta_n = \frac{1}{n^p} \) for some \( p > 0 \) or \( \beta_n = \gamma^n \) where \(|\gamma| < 1\) if it works and \( K > 0 \). For example,
\[ \beta_n = \frac{1}{n^2}, \quad \beta_n = \left(\frac{1}{2}\right)^n. \]

**Example** Find the rates of convergence of the sequences: \( \left\{ \frac{n+3}{n+7} \right\} \) and \( \left\{ \frac{2^n+3}{2^n+7} \right\} \).

Clearly, both converge to 1. Which one converges to 1 faster? Let us first check out the convergence numerically:

\[
\begin{array}{c|c|c}
\text{n} & \left\{ \frac{n+3}{n+7} \right\} & \left\{ \frac{2^n+3}{2^n+7} \right\} \\
\hline
1 & 0.50000000000000 & 0.55555555555556 \\
2 & 0.55555555555556 & 0.63636363636364 \\
3 & 0.60000000000000 & 0.73333333333333 \\
4 & 0.63636363636364 & 0.82608695652174 \\
5 & 0.66666666666667 & 0.94366197183099 \\
6 & 0.69230769230769 & 0.99743589743590 \\
7 & 0.71428571428571 & 0.99929287090559 \\
8 & 0.73333333333333 & 0.99961202715809 \\
9 & 0.75000000000000 & 0.99992928709059 \\
10 & 0.76470588235294 & 0.99999929287091 \\
\end{array}
\]

Clearly from the numerical results, we see \( \left\{ \frac{2^n+3}{2^n+7} \right\} \) converges to 1 much faster than \( \left\{ \frac{n+3}{n+7} \right\} \).

Now let us determine the rate of convergence for each sequence algebraically by definition.

For the sequence \( \left\{ \frac{n+3}{n+7} \right\} \), observe that

\[
\left| x_n - x^* \right| = \left| \frac{n+3}{n+7} - 1 \right| = \left| \frac{-4}{n+7} \right| = \frac{4}{n+7}.
\]

Choose \( \{\beta_n\} = \left\{ \frac{1}{n} \right\} \).

\[
\lim_{n \to \infty} \frac{\left| x_n - x^* \right|}{\beta_n} = \lim_{n \to \infty} \frac{\left| \frac{n+3}{n+7} - 1 \right|}{\frac{1}{n}} = \lim_{n \to \infty} \frac{4}{n+7} \cdot \frac{1}{n} = \frac{4n}{n+7} = 4. \quad K = 4.
\]

For the sequence \( \left\{ \frac{2^n+3}{2^n+7} \right\} \), observe that

\[
\left| x_n - x^* \right| = \left| \frac{2^n+3}{2^n+7} - 1 \right| = \left| \frac{-4}{2^n+7} \right| = \frac{4}{2^n+7}.
\]

Choose \( \{\beta_n\} = \left\{ \frac{1}{2^n} \right\} \).

\[
\lim_{n \to \infty} \frac{\left| \frac{2^n+3}{2^n+7} - 1 \right|}{\beta_n} = \lim_{n \to \infty} \frac{\left| \frac{2^n+3}{2^n+7} - 1 \right|}{\frac{1}{2^n}} = \lim_{n \to \infty} \frac{4(2^n)}{2^n+7} = 4. \quad K = 4.
\]

Hence, \( \frac{n+3}{n+7} = 1 + O\left(\frac{1}{n}\right) \) and \( \frac{2^n+3}{2^n+7} = 1 + O\left(\frac{1}{2^n}\right) \), that is, \( \frac{n+3}{n+7} \to 1 \) linearly as \( \frac{1}{n} \to 0 \) and \( \frac{2^n+3}{2^n+7} \to 1 \) exponentially as \( \frac{1}{2^n} \to 0 \).

**Example** Find the limit of each sequence and determine its rate of convergence.

i. \( \left\{ \frac{1-2n^2}{3n^2+n-1} \right\} \)  \quad ii. \( \left\{ \frac{\sqrt{1+2n^2}}{3n} \right\} \)  \quad iii. \( \{\sin\left(\frac{1}{n}\right)\} \)  \quad iv. \( \{\ln\left(\frac{2n-1}{2n+1}\right)\} \)

i. \( \left\{ \frac{1-2n^2}{3n^2+n-1} \right\} \)
\[
\lim_{n \to \infty} \frac{1 - 2n^2}{3n^2 + n - 1} = -\frac{2}{3}
\]

\[
\frac{\left| \frac{1 - 2n^2}{3n^2 + n - 1} - \left( -\frac{2}{3} \right) \right|}{\left| \frac{1 + 2n}{3(3n^2 + n - 1)} \right|} = \frac{1 + 2n}{3(3n^2 + n - 1)}
\]

Let \( \{ \beta_n \} = \{ \frac{1}{n} \} \).

\[
\lim_{n \to \infty} \frac{1 - 2n^2}{3n^2 + n - 1} = 2 = K
\]

\[
\frac{1}{3n^2 + n - 1} = -\frac{2}{3} + O\left( \frac{1}{n} \right)
\]

ii.

\[
\lim_{n \to \infty} \frac{\sqrt{1 + 2n^2} - \sqrt{2}}{3n} = \lim_{n \to \infty} \frac{1 + 2n^2}{9n^2} = \frac{2}{9} = \frac{\sqrt{2}}{3}
\]

\[
\frac{\sqrt{1 + 2n^2}}{3n} - \frac{\sqrt{2}}{3} = \frac{\sqrt{1 + 2n^2} - n\sqrt{2}}{3n} = \frac{\sqrt{1 + 2n^2} - \sqrt{2n^2}}{3n} \left( \frac{\sqrt{1 + 2n^2} + \sqrt{2n^2}}{\sqrt{1 + 2n^2} + \sqrt{2n^2}} \right)
\]

\[
= \frac{1 + 2n^2 - 2n^2}{3n(\sqrt{1 + 2n^2} + \sqrt{2n^2})} = \frac{1}{3n(\sqrt{1 + 2n^2} + \sqrt{2n^2})}
\]

Let \( \{ \beta_n \} = \{ \frac{1}{n^2} \} \).

\[
\lim_{n \to \infty} \frac{\frac{\sqrt{1 + 2n^2}}{3n} - \frac{\sqrt{2}}{3}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{1}{3n(\sqrt{1 + 2n^2} + \sqrt{2n^2})}
\]

\[
= \lim_{n \to \infty} \frac{n^2}{3n(\sqrt{1 + 2n^2} + \sqrt{2n^2})} = \frac{1}{6\sqrt{2}} = K
\]

\[
\frac{\sqrt{1 + 2n^2}}{3n} = \frac{\sqrt{2}}{3} + O\left( \frac{1}{n^2} \right)
\]

iii.

\[
\lim_{n \to \infty} \sin\left( \frac{1}{n} \right) = 0
\]

It is know that \( \sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \ldots \).

\[
|\sin\left( \frac{1}{n} \right) - 0| = \left| \frac{1}{n} - \frac{1}{3!} \left( \frac{1}{n} \right)^3 + \frac{1}{5!} \left( \frac{1}{n} \right)^5 - \ldots \right|
\]

Let \( \{ \beta_n \} = \{ \frac{1}{n} \} \).

\[
\lim_{n \to \infty} \left| \frac{\sin\left( \frac{1}{n} \right)}{\frac{1}{n}} - 0 \right| = \lim_{n \to \infty} \left| \frac{1}{n} - \frac{1}{3!} \left( \frac{1}{n} \right)^3 + \frac{1}{5!} \left( \frac{1}{n} \right)^5 - \ldots \right|
\]

\[
= \lim_{n \to \infty} \left| 1 - \frac{1}{3!} \left( \frac{1}{n} \right)^2 + \frac{1}{5!} \left( \frac{1}{n} \right)^4 - \ldots \right| = 1 = K
\]

\[
\sin\left( \frac{1}{n} \right) = 0 + O\left( \frac{1}{n^2} \right)
\]
iv. \[
\lim_{n \to \infty} \ln \left( \frac{2n - 1}{2n + 1} \right) = \ln(1) = 0
\]

It is know that \( \ln(1 + x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \ldots \)

\[
\left| \ln \left( \frac{2n - 1}{2n + 1} \right) - 0 \right| = \left| \ln \left( 1 + \frac{-2}{2n + 1} \right) \right| = \left| -\frac{2}{2n + 1} - \frac{1}{2} \left( \frac{-2}{2n + 1} \right)^2 + \frac{1}{3} \left( \frac{-2}{2n + 1} \right)^3 - \ldots \right|
\]

Let \( \{ \beta_n \} = \left\{ \frac{1}{n} \right\} \).

\[
\lim_{n \to \infty} \left| \ln \left( \frac{n-1}{2n+1} \right) \right| = \lim_{n \to \infty} \left| \frac{-\frac{2}{2n+1} - \frac{1}{2} \left( \frac{-2}{2n+1} \right)^2 + \frac{1}{3} \left( \frac{-2}{2n+1} \right)^3 - \ldots}{\frac{1}{n}} \right|
\]

\[
= \lim_{n \to \infty} \left| -\frac{2n}{2n+1} - n \left( \frac{-2}{2n+1} \right)^2 + n \left( \frac{-2}{2n+1} \right)^3 - \ldots \right| = 1 = K
\]

\[
\ln \left( \frac{2n - 1}{2n + 1} \right) = O \left( \frac{1}{n} \right)
\]

b. Order of Convergence:

**Definition** Suppose \( \{ x_n \}_{n=0}^{\infty} \) is a sequence that converges to \( x^* \) with \( x_n \neq x^* \) for all \( n \). The sequence \( \{ x_n \}_{n=1}^{\infty} \) is said to converge to \( x^* \) of order of \( a \), with asymptotic error constant \( \lambda \) if there exist positive constants \( \lambda \) and \( a \) such that

\[
\lim_{n \to \infty} \left| \frac{x_{n+1} - x^*}{x_n - x^*} \right|^a = \lambda.
\]

If \( a = 1 \), then we say \( x_n \) converges to \( x^* \) linearly.

If \( a = 2 \), then we say \( x_n \) converges to \( x^* \) quadratically.

If \( 0 < a < 1 \), then we say \( x_n \) converges to \( x^* \) sublinearly.

If \( 1 < a < 2 \), then we say \( x_n \) converges to \( x^* \) superlinearly.

Note that:

- The larger the value of \( a \) is, the faster \( x_n \) converges to \( x^* \); and the smaller the value of \( \lambda \) is, the faster \( x_n \) converges to \( x^* \).

The reason is the following. For large \( n \),

\[
\left| \frac{x_{n+1} - x^*}{x_n - x^*} \right|^a \approx \lambda, \quad \left| \frac{x_{n+1} - x^*}{x_n - x^*} \right|^a \approx \lambda \left| x_n - x^* \right|^a, \quad \text{let } m > n,
\]

\[
\left| x_{m} - x^* \right| \approx \lambda \left| x_{m-1} - x^* \right|^a \approx \lambda \left( \lambda \left| x_{m-2} - x^* \right|^a \right)^a = \lambda^{a+1} \left| x_{m-2} - x^* \right|^{a^2}
\]

\[
\approx \lambda^{a+1} \left( \lambda \left| x_{m-3} - x^* \right|^a \right)^{a^2} = \lambda^{a^2+a+1} \left| x_{m-3} - x^* \right|^{a^3} \approx \ldots
\]

\[
\approx \lambda^{a^{m-1} + \ldots + a^2 + a + 1} \left| x_n - x^* \right|^{a^{m-a}} = \lambda^{\frac{a^{m-a} - 1}{a-1}} \left| x_n - x^* \right|^{a^{m-a}}
\]

- If \( \lim_{n \to \infty} \left| \frac{x_{n+1} - x^*}{x_n - x^*} \right| = 0 \) but \( \lim_{n \to \infty} \left| \frac{x_{n+1} - x^*}{x_n - x^*} \right|^2 = \infty \), then \( x_n \) converges to \( x^* \) superlinearly.
The reason is the following. Observe that $\lim_{n \to \infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|} = 0$ implies

$$|x_{n+1} - x^*| = C |x_n - x^*| |x_n - x^*|^{\gamma}$$

for some $0 < \gamma < 1$ and $C > 0$. Hence,

$$\lim_{n \to \infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^{1+\gamma}} = \lim_{n \to \infty} C |x_n - x^*|^{1+\gamma} = C$$

**Example** Determine the order of convergence (1 or 2) and the asymptotic error constant $\lambda$ of the given sequence.

a. $\left\{\frac{1}{5^n}\right\}_{n=1}^{\infty}$

b. $\left\{\frac{1}{5^{2n}}\right\}_{n=1}^{\infty}$

c. $\left\{\frac{1}{5^{2n}}\right\}_{n=1}^{\infty}$

d. $\left\{\frac{1}{n^n}\right\}_{n=1}^{\infty}$

a. $\lim_{n \to \infty} \frac{1}{5^n} = 0$, $L = 0$.

$$\lim_{n \to \infty} \frac{|x_{n+1} - L|}{|x_{n} - L|} = \lim_{n \to \infty} \frac{\frac{1}{5^{n+1}}}{\frac{1}{5^n}} = \frac{1}{5}.$$  

The order of convergence is 1 and the asymptotic error constant is $\frac{1}{5}$.

b. $\lim_{n \to \infty} \frac{1}{5^{2n}} = 0$, $x^* = 0$.

$$\lim_{n \to \infty} \frac{|x_{n+1} - x^*|}{|x_{n} - x^*|} = \lim_{n \to \infty} \frac{\frac{1}{5^{2(n+1)}}}{\frac{1}{5^{2n}}} = \lim_{n \to \infty} \frac{\frac{1}{5^{2n+2}}}{\frac{1}{5^{2n}}} = \frac{1}{5^2} = \frac{1}{25}.$$  

The order of convergence is 1 and the asymptotic error constant is $\frac{1}{25}$.

c. $\lim_{n \to \infty} \frac{1}{5^{2n}} = 0$, $x^* = 0$.

$$\lim_{n \to \infty} \frac{|x_{n+1} - x^*|}{|x_{n} - x^*|} = \lim_{n \to \infty} \frac{\frac{1}{5^{2n+1}}}{\frac{1}{5^{2n}}} = \lim_{n \to \infty} \frac{\frac{5^2}{5^{2n+1}}}{\frac{5^2}{5^{2n}}} = \lim_{n \to \infty} \frac{5^2}{5^2} = \frac{5^2}{5^2} = 0$$

$$\lim_{n \to \infty} \frac{|x_{n+1} - x^*|^2}{|x_{n} - x^*|^2} = \lim_{n \to \infty} \frac{\frac{1}{5^{2n+1}}}{\left(\frac{1}{5^{2n}}\right)^2} = \lim_{n \to \infty} \frac{\frac{5^{2n+1}}{5^{2n}}}{\left(\frac{1}{5^{2n}}\right)^2} = \frac{5^{2n+1}}{1} = 1$$

The order of convergence is 2 and the asymptotic error constant is 1.

d. $\lim_{n \to \infty} \frac{1}{n^n} = 0$, $x^* = 0$. 

5
\[
\lim_{n \to \infty} \frac{x_{n+1} - x^*}{x_n - x^*} = \lim_{n \to \infty} \frac{1}{n+1} = \lim_{n \to \infty} \frac{n}{(n+1)^{n+1}} = \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^n \frac{1}{(n+1)^n} = 0
\]

\[
\lim_{n \to \infty} \frac{x_{n+1} - x^*}{(x_n - x^*)^2} = \lim_{n \to \infty} \frac{1}{n+1} = \lim_{n \to \infty} \frac{n}{(n+1)^{n+1}} = \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^n \frac{1}{(n+1)^n} = \infty
\]

The order of convergence is 1 and the asymptotic error constant is 0 so \( \frac{1}{n^a} \to 0 \) superlinearly.

**Example** Find the order of convergence and the asymptotic error constant of the sequence \( \left\{ \frac{1}{2} \left( x_n + \frac{a}{x_n} \right) \right\} \) where \( a > 0 \).

We know \( x_n \to \sqrt{a} \).

\[
x_{n+1} - \sqrt{a} = \frac{1}{2} \left( x_n + \frac{a}{x_n} \right) - \sqrt{a} = \frac{1}{2} x_n^2 + a - 2\sqrt{a} x_n = \frac{1}{2} x_n^2 (x_n - \sqrt{a})^2.
\]

Then

\[
\lim_{n \to \infty} \frac{x_{n+1} - \sqrt{a}}{(x_n - \sqrt{a})^2} = \lim_{n \to \infty} \frac{1}{2} \left( x_n + \frac{a}{x_n} \right) - \sqrt{a} = \lim_{n \to \infty} \frac{1}{2} x_n (x_n - \sqrt{a})^2
\]

\[
= \lim_{n \to \infty} \frac{1}{2 x_n} = \frac{1}{2\sqrt{a}}.
\]

Hence, \( x_n \to \sqrt{a} \) quadratically with asymptotic error constant \( \lambda = \frac{1}{2\sqrt{a}} \).

**Exercises:**

1. Consider real sequences: \( \{x_n\}, \{y_n\}, \{z_n\}, \{u_n\} \) and \( \{w_n\} \). Suppose that they all converge to a real number \( L \).
   a. Suppose that
      \[
x_n = L + O\left( \frac{1}{\sqrt{n}} \right), \quad y_n = L + O\left( \frac{1}{n} \right), \quad z_n = L + O\left( \frac{1}{3^n} \right), \quad u_n = L + O\left( \frac{1}{e^n} \right) \quad \text{and} \quad w_n = L + O\left( \frac{1}{4^n} \right).
      \]
      Rank the sequences in an increasing order, based on their rates of convergence.
   b. Suppose we also know that
      \[
      \lim_{n \to \infty} \frac{|x_{n+1} - L|}{|x_n - L|} = 0.95, \quad \lim_{n \to \infty} \frac{|y_{n+1} - L|}{|y_n - L|} = 0.85, \quad \lim_{n \to \infty} \frac{|z_{n+1} - L|}{|z_n - L|} = 0.9,
      \]
      \[
      \lim_{n \to \infty} \frac{|u_{n+1} - L|}{|u_n - L|^{1/2}} = 0.9, \quad \lim_{n \to \infty} \frac{|w_{n+1} - L|}{|w_n - L|^{1.5}} = 0.9.
      \]
      Assume that the initial error \( |p_0 - L| \) for each sequence is 1 (that is \( |x_0 - L| = 1, \quad |y_0 - L| = 1, \ldots \)), give a rough estimate (meaning for \( n \geq 0 \), assume that
      \[
      \frac{|x_{n+1} - L|}{|x_n - L|} \approx 0.95, \quad \frac{|y_{n+1} - L|}{|y_n - L|} \approx 0.85, \quad \ldots \)
      for the error \( |p_{6} - L| \) for each sequence (that is \( |x_6 - L|, \quad |y_6 - p| \ldots \)).
Rank the sequences in an increasing order, based on their order of convergence.

2. For each sequence \( \langle x_n \rangle \), (i) find the limit \( x^* \); (ii) determine the rate of convergence.
   (a) \( \left\{ \frac{n - 1}{n^2 + 2} \right\} \)  
   (b) \( \left\{ \frac{n^2 - 1}{2n^2 + n + 1} \right\} \)  
   (c) \( \left\{ \sqrt{n + 1} - \sqrt{n} \right\} \)  
   (d) \( \left\{ \frac{1}{n!} \right\} \)

3. Show that the convergence of the sequence \( \langle x_{n+1} \rangle \) where \( x_{n+1} = \frac{x_n^3 + 3x_n a}{3x_n^2 + a} \) where \( a > 0 \) toward \( \sqrt[3]{a} \) is 3rd order and find the asymptotic error constant \( \lambda \).

4. Let \( a \neq 0 \) and \( 0 < x_0 < \frac{2}{a} \). Find the order of convergence of the sequence \( \langle x_{n+1} \rangle \) where \( x_{n+1} = x_n(2 - ax_n) \) toward \( \frac{1}{a} \) and find the asymptotic error constant \( \lambda \).