**Intermediate Value Theorem**

**Theorem:** If \( f \) is continuous on \([a, b]\) and \( K \) is a number between \( f(a) \) and \( f(b) \), then there exists a number \( c \) in \((a, b)\) for which \( f(c) = K \).

**Notes:**

(i) If \( f(a)f(b) < 0 \), then either \( f(a) \) or \( f(b) \) is less than 0 \((K = 0)\), and there exists a number \( c \) in \((a, b)\) for which \( f(c) = 0 \). The number \( c \) is a solution of the equation \( f(x) = 0 \).

(ii) The condition in (i) is a sufficient condition, that is, if \( f(a)f(b) > 0 \), it is still possible to have a zero in \((a, b)\).

**Examples:**

\(-2 \leq x \leq 2\)

(a) one zero

(b) two

(c) no zero


**Bisection Method**

**Algorithm:** For $k = 1, 2, \ldots$, having $f(a_k)f(b_k) < 0$, compute $x_k = \frac{1}{2}(a_k + b_k)$ and

$$
\begin{cases}
    a_{k+1} = a_k & \text{if } f(a_k)f(x_k) < 0 \text{ or } \\
    b_{k+1} = x_k
\end{cases}
\quad \text{or} \quad
\begin{cases}
    a_{k+1} = x_k & \text{if } f(x_k)f(b_k) < 0. \\
    b_{k+1} = b_k
\end{cases}
$$

The algorithm stops if $f(x_k) = 0$ or $|f(x_k)| < \varepsilon$ and $x^* \approx x_k$. 
Convergence

**Theorem:** Let $f$ be continuous on $[a, b]$, and $f(a)f(b) < 0$. Let $x^*$ be the unique solution of the equation $f(x) = 0$ for $x$ in $[a, b]$, and $\{x_n\}$ be generated by the Bisection Method. Then

$$\lim_{n \to \infty} x_n = x^*.$$

$$|x_1 - x^*| \leq \frac{1}{2}(b-a)$$

$$|x_2 - x^*| \leq \frac{1}{2}\left(\frac{1}{2}(b-a)\right) = \frac{1}{2^2}(b-a)$$

$$\vdots$$

$$|x_n - x^*| \leq \frac{1}{2^n}(b-a).$$

Hence,

$$\lim_{n \to \infty} |x_n - x^*| \leq \lim_{n \to \infty} \frac{1}{2^n}(b-a) = 0 \Rightarrow \lim_{n \to \infty} x_n = x^*.$$
Theorem: Let \( \{ x_n \} \) be generated by the Bisection Method and \( \lim_{n \to \infty} x_n = x^* \). Then
\[
x_n = x^* + O\left(\frac{1}{2^n}\right).
\]

Note: For a given \( \varepsilon > 0 \), \( |x_N - x^*| < \varepsilon \) if
\[
N > \frac{\ln(b - a) - \ln \varepsilon}{\ln 2}.
\]
Let \( N = \left\lceil \frac{\ln(b - a) - \ln \varepsilon}{\ln 2} \right\rceil \).