Rolle’s and Mean Value Theorems

Rolle’s Theorem: Suppose that \( f \) is continuous on \([a, b]\) and is differentiable on \((a, b)\). If \( f(a) = f(b) \), then there exists a number \( c \) in \((a, b)\) such that \( f'(c) = 0 \).

Mean Value Theorem: Suppose that \( f \) is continuous in \([a, b]\) and is differentiable on \((a, b)\). Then there exists a number \( c \) in \((a, b)\) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

Notes:

- If \( f'(x) \neq 0 \) for all \( x \) in \((a, b)\), then \( f(x_1) \neq f(x_2) \) (\( = 0 \)).
- When \( b \) is close to \( a \) (\(|b - a|\) is small), \( f'(x) \approx \frac{f(b) - f(a)}{b - a} \) for \( x \in (a, b) \).
Fixed-Point of a Function:

**Definition:** A number $p$ is said to be a **fixed point** of a function $g(x)$ if $g(p) = p$.

$e^{-p} = p$

$y = x^2 + 1$

$y = x + \cos(x)$
Existence and Uniqueness:

**Theorem:** Let \( g \) be continuous on \([a, b]\).

i. If \( a \leq g(x) \leq b \) for all \( x \) in \([a, b]\), then \( g(x) \) has a fixed point \( p \) in \([a, b]\).

ii. If, in addition, \( g'(x) \) exists on \((a, b)\) and there exists a constant \( 0 < K < 1 \) such that

\[
|g'(x)| \leq K \text{ for all } x \text{ in } (a, b),
\]

then \( p \) is unique.

**Note that:**

- Both conditions: \( a \leq g(x) \leq b \) for all \( x \) in \([a, b]\) and
  \[
  |g'(x)| \leq K \text{ for all } x \text{ in } (a, b)
  \]
  are **sufficient conditions**.

- Because \( g'(x) \) is the slope of the tangent line to the curve \( y = g(x) \) at \( x \), \( |g'(x)| \leq K < 1 \) means that the graph of \( g(x) \) does not grow as faster than \( y = x \) and not slower than \( y = -x \).
The Fixed-Point Algorithm:
Assume that \( g(x) \) has a unique fixed point \( p \) in \([a, b]\).
The Fixed-Point Algorithm is an algorithm that finds the fixed-point \( p \) of \( g(x) \) in \([a, b]\).

Algorithm: Given \( g(x) \), and \([a, b]\), choose \( p_0 \) in \([a, b]\) and compute \( p_1, p_2, \ldots \), as follows:

\[
p_n = g(p_{n-1}) \quad \text{for } n = 1, 2, \ldots
\]

Implement the algorithm in a programming language which does the following: Input \( g(x) \), interval \([a, b]\), \( p_0 \) in \([a, b]\), \( \epsilon \) and \( K_{\text{max}} \), and compute \( p_n = g(p_{n-1}) \) for \( n = 1, 2, \ldots \). The program terminates if

i. \( |p_n - p_{n-1}| < \epsilon \) and then \( p \approx p_n \); or

ii. \( p_n > b \) or \( p_n < a \), and the program fails; or

iii. \( n = K_{\text{max}} \).
The MatLab program fixpt.m implements the Fixed-Point Algorithm to find $p_n$ with input
(1) the function $gfun$ for $g(x)$;
(2) an initial approximation $p_0$ to the fixed-point;
(3) an accuracy requirement $\varepsilon$; and
(4) the maximum number for iterations $K_{\text{max}}$.

**Graphically:**

**Numerically:**

$$g(x) = \frac{1}{3} \left( 2 - e^x + x^2 \right)$$

$$p_0 = 0$$

$$p_1 = g(p_0) = g(0) = \frac{1}{3}$$

$$p_2 = g(p_1) = g\left(\frac{1}{3}\right)$$

$$= \frac{1}{3} \left( 2 - e^{1/3} + \frac{1}{9} \right)$$

$$= 0.238499562$$
Convergence

Questions: Assume that $g$ has a unique fixed point $p$ in $[a, b]$ and $p_0$ is in $[a, b]$. Let $p_n = g(p_{n-1}), \ n = 1, 2, \ldots$.

(1) Under what condition(s), does $p_n$ converge to $p$?

(2) If $\lim_{n \to \infty} p_n = p$, what is the rate of converge or the order of convergence?

Fixed-Point Theorem:

Let $g$ be continuous on $[a, b]$ and $a \leq g(x) \leq b$.

Suppose that $g'(x)$ exists for all $x$ in $(a, b)$, and

$$|g'(x)| \leq K \text{ for all } x \text{ in } (a, b) \text{ where } 0 < K < 1.$$ 

Then $\lim_{n \to \infty} p_n = p$ for any $p_0$ in $[a, b]$, and

$$|p_n - p| \leq K^n \max\{p_0 - a, \ b - p_0\} \text{ and}$$

$$|p_n - p| \leq \frac{K^n}{1 - K} \ |p_1 - p_0|, \text{ for all } n = 1, 2, \ldots.$$