2.4 - Convergence of the Newton Method and Modified Newton Method

Newton Method:

Given \( f(x), x_0 \) in \([a, b]\) and \( \varepsilon \), for \( n = 1, 2, \ldots \),

(1) compute \( x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \);

(2) if \( |x_n - x_{n-1}| < \varepsilon \) or \( |f(x_n)| < \varepsilon \), then the algorithm is terminated and \( x^* \approx x_n \); otherwise goto Step (1).

We use the Fixed Point Theorem to show the convergence of the Newton Method:

Define: \( g_{\text{newton}}(x) = x - \frac{f(x)}{f'(x)} \).
Zeros of a function:

Recall: Let \( f(r) = 0 \). \( x = r \) is called a zero or a root of \( f(x) \).

1. If \( x = r \) is a simple zero of \( f(x) \), then there exists \( Q(x) \) such that

\[
f(x) = (x - r)Q(x) \quad \text{and} \quad \lim_{x \to r} Q(x) \neq 0.
\]

Example: \( f(x) = x^3 - 8 = (x - 2)(x^2 + 2x + 4) \). \( r = 2 \) and \( Q(x) = x^2 + 2x + 4 \).

Example: \( f(x) = \sin(x) = x \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots\right) \).

\( r = 0, \ Q(x) = \frac{\sin(x)}{x}, \ \lim_{x \to 0} Q(x) = 1 \)

2. If \( x = r \) is a zero of multiplicate \( m \), then there exists \( Q(x) \) such that

\[
f(x) = (x - r)^m Q(x) \quad \text{and} \quad \lim_{x \to r} Q(x) \neq 0.
\]

Example: \( f(x) = x^3 - 2x^2 + x = x(x - 1)^2 \), \( x = 0 \) is a simple zero and \( x = 1 \) is a zero of multiplicity 2.
Example: $f(x) = \cos(x) - 1 = x^2 \left( \frac{1}{2!} - \frac{x^2}{4!} + \ldots \right)$. $x = 0$ is a zero of multiplicity 2 and $Q(x) = \frac{\cos(x) - 1}{x^2}$.

(3) Let $x = r$ be a zero of multiplicity $m$. Then

$$f(r) = 0, f'(r) = 0, \ldots, f^{(m-1)}(r) = 0 \text{ but } f^{(m)}(r) \neq 0$$

Example: $f(x) = \sin(x)$, $r = 0$, $m = 1$
Example: $f(x) = e^x - 1 - x$, $r = 0$, $m = 2$
Local Convergence Property of Newton’s Method:

\[ g_{\text{newton}}(x) = x - \frac{f(x)}{f'(x)} \]

Case 1. \( f(x^*) = 0 \) but \( f'(x^*) \neq 0 \) (\( x^* \) is a simple zero)

**Theorem:** Let \( f, f' \) and \( f'' \) be continuous in \([a, b]\). If \( x^* \) is in \([a, b]\) and \( f'(x^*) \neq 0 \), then there exists a \( \delta > 0 \) such that \( \{x_n\} \) generated by Newton’s method converges to \( x^* \) for any \( x_0 \) in \([x^* - \delta, x^* + \delta]\).

**Note that:**

a. Since the convergence of Newton method depends on \( \delta \), \( \lim_{n \to \infty} x_n = x^* \) only locally.

b. By the Fixed-Point Theorem, \( x_n = x^* + O(K^n) \) where

\[
|g_{\text{newton}}(x)| = \left| \frac{f(x)f''(x)}{[f'(x)]^2} \right| \leq K < 1 \text{ for all } x \text{ in } [x^* - \delta, x^* + \delta].
\]
Example: (1) $f(x) = x^2 - 2$  (2) $f(x) = x^3 + x + 1$
Modified Fixed Point Theorem:
Let \( p \) be a fixed point of \( g(x) \). If \( g'(p) = 0 \) and \( g''(p) \neq 0 \), then
\( p_n \rightarrow p \) quadratically \((\alpha = 2)\) with an asymptotic error constant
\( \lambda = \frac{1}{2}|g''(p)|. \)

Order of Convergence of Newton’s Method: (Case 1.)
Let \( \{x_n\} \) be generated by the Newton Method. If \( x^* \) is a simple
zero of \( f(x) \), then \( x_n \rightarrow x^* \) quadratically with an asymptotic error
constant \( \lambda = \left| \frac{f^{''}(x^*)}{2f'(x^*)} \right|. \)
Example: \( f(x) = x^2 - 2. \quad f(x) = x^3 + x + 1. \)
Summary:
Review: $x_n \to x^*$ in an order $\alpha$ with the asymptotic error constant $\lambda$ if
$$\lim_{n \to \infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^{\alpha}} = \lambda.$$

(1) Order of convergence of the **Fixed Point Method**
(Algorithm): Solve $g(p) = p$ for $p$.
Let $p_{n+1} = g(p_n)$ for $n = 0, 1, \ldots$, for a given $p_0$.

<table>
<thead>
<tr>
<th>Case</th>
<th>$\alpha$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g'(p) \neq 0$</td>
<td>1</td>
<td>$</td>
</tr>
<tr>
<td>$g'(p) = 0$ but $g''(p) \neq 0$</td>
<td>2</td>
<td>$\frac{1}{2}</td>
</tr>
</tbody>
</table>

Since we do not know $p$, we usually estimate an upper bound of $\lambda$ by estimating $K$ over an interval such that
$$|g'(x)| \leq K \text{ for all } x \in [a, b] \quad (\text{or } \frac{1}{2}|g''(p)| \leq K \text{ in the 2nd case}).$$
(2) Order of convergence of the **Newton Method**: Solve \( f(x) = 0 \) for \( x \).

Let \( f(x^*) = 0 \).

Let \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = g_{\text{newton}}(x_n) \) for \( n = 0, 1, \ldots \), for a given \( x_0 \) near \( x^* \).

<table>
<thead>
<tr>
<th>Case</th>
<th>( \alpha )</th>
<th>( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(x^<em>) \neq 0 ) (( x^</em> ) is a simple zero)</td>
<td>2</td>
<td>( \frac{1}{2} \left</td>
</tr>
<tr>
<td>( f^{(k)}(x^<em>) = 0 ) for ( k = 0, 1, \ldots, m - 1 ) but ( f^{(m)}(x^</em>) \neq 0 ) (( x^* ) is a zero of multiplicity ( m ))</td>
<td>1</td>
<td>( \frac{m - 1}{m} )</td>
</tr>
</tbody>
</table>
(3) Order of convergence of the **Modified Newton Method**:
Solve $f(x) = 0$ for $x$.
Let $x = x^*$ be the zero of multiplicity $m$.

(i) $x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)} = g_{MN1}(x_n)$ for $n = 0, 1, \ldots$, for a given $x_0$ near $x^*$.

(ii) $x_{n+1} = x_n - \frac{\mu(x_n)}{\mu'(x_n)} = g_{MN2}(x_n)$ for $n = 0, 1, \ldots$, for a given $x_0$ near $x^*$ where $x = x^*$ is a simple zero of $\mu(x)$

$\alpha = 2$ and $\lambda = \frac{1}{2} \left| g''_{MN1}(x^*) \right|$

Example: $f(x) = \ln(x + 1) - x$, $x^* = 0$, $m = 2$
f(x) = \ln(x+1) - x, Newton Method

f(x) = \ln(x+1) - x, Modified Newton Method (i)

f(x) = \ln(x+1) - x, Modified Newton Method (ii)