In Chapter 5, we study the numerical methods for solving initial-value problems for ordinary differential equations:

\[ y'(t) = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha. \]

1. **Lipschitz Condition:**

**Definition 5.1** A function \( f(t, y) \) is said to satisfy a **Lipschitz condition** in the variable \( y \) on a set \( D \subset \mathbb{R}^2 \) if there exists a constant \( L > 0 \) such that

\[ |f(t_1, y_1) - f(t_2, y_2)| \leq L \|y_1 - y_2\|, \]

whenever both points \((t, y_1)\) and \((t, y_2)\) are in \( D \). The constant \( L \) is called a **Lipschitz constant** for \( f \).

**Example** Let \( f(t, y) = 1 + ty^2 \) and \( D = \{(t, y) \mid 0 \leq t \leq 2, \quad -1 \leq y \leq 1\} \). Does \( f \) satisfy a Lipschitz condition on \( D \)? If so, find its Lipschitz constant.

Let \((t, y_1)\) and \((t, y_2)\) be in \( D \), i.e., \( t \) is in \([0, 2]\) and \( y \) is in \([-1, 1]\). Observe that

\[ |f(t, y_1) - f(t, y_2)| = |1 + ty_1^2 - (1 + ty_2^2)| = |t||y_1^2 - y_2^2| = |t||y_1 + y_2||y_1 - y_2|. \]

Because \( |t| \leq 2 \) and \( |y_1 + y_2| \leq 2 \), we have

\[ |f(t, y_1) - f(t, y_2)| \leq (2)(2)|y_1 - y_2| = 4|y_1 - y_2|. \]

So, \( f \) satisfies a Lipschitz condition and its Lipschitz constant is 4.

Note that the Lipschitz constant \( L \) is not unique, that is, for any \( \overline{L} > 4 \), the inequality

\[ |f(t, y_1) - f(t, y_2)| \leq \overline{L}|y_1 - y_2| \]

also holds. So, in practice, we want to find a \( L \) as small as possible.

2. **Convex Set:**

**Definition 5.2** A set \( D \subset \mathbb{R}^2 \) is said to be **convex** if whenever \((t_1, y_1)\) and \((t_2, y_2)\) belong to \( D \) and \( \lambda \) is in \([0, 1]\), the point \((1-\lambda)t_1 + \lambda t_2, \quad (1-\lambda)y_1 + \lambda y_2\) also belongs to \( D \).

Graphically, it is easy to see if a given set \( D \) is convex by checking if the line segment between points \((t_1, y_1)\) and \((t_2, y_2)\) contain completely in \( D \).

**Example:**

- a. and c. are convex and
- b. and d. are not.
3. A Sufficient Condition for Lipschitz Condition:

**Theorem 5.3** Suppose $f(t,y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If a constant $L > 0$ exists with

$$\left|\frac{\partial f}{\partial y}(t, y)\right| \leq L, \text{ for all } (t, y) \in D,$$

then $f$ satisfies a Lipschitz condition on $D$ in the variable $y$ with Lipschitz constant $L$.

**Example** Let $f(t,y) = 1 + ty^2$ and $D = \{(t,y) | 0 \leq t \leq 2, \ -1 \leq y \leq 1\}$. Does $f$ satisfy a Lipschitz condition on $D$? If so, find its Lipschitz constant.

The partial derivative of $f$ with respect to $y$ is $\frac{\partial f}{\partial y}(t, y) = 2ty$. Because

$$\left|\frac{\partial f}{\partial y}(t, y)\right| = |2ty| = 2|t||y| \leq 2(2)(1) = 4,$$

$f$ satisfies a Lipschitz condition with Lipschitz constant 4.

**Example** Let $f(t,y) = 1 + t\sin(ty)$. Determine if $f$ satisfies a Lipschitz condition in $D = \{(t,y) | 0 \leq t \leq 2, \ -\infty < y < \infty\}$.

Compute the partial derivative of $f$ with respect to $y$:

$$\frac{\partial f}{\partial y}(t, y) = t\cos(ty) = t^2 \cos(ty).$$

Because

$$|\cos(ty)| \leq 1 \text{ and } |t| \leq 2,$$

$$\left|\frac{\partial f}{\partial y}(t, y)\right| = |t^2 \cos(ty)| = |t^2||\cos(ty)| \leq |t^2| \leq 4.$$

So, $f$ satisfies a Lipschitz condition with a constant 4.

4. A Sufficient Condition for the Uniqueness of Solution of an Initial Value Problem:

**Theorem 5.4** Suppose that $D = \{(t, y) | a \leq t \leq b, \ -\infty < y < \infty\}$ and that $f(t,y)$ is continuous on $D$. If $f$ satisfies a Lipschitz condition on $D$ in the variable $y$, then the initial-value problem

$$y'(t) = f(t, y), \ a \leq t \leq b, \ y(a) = \alpha$$

has a unique solution $y(t)$ for $a \leq t \leq b$.

**Example** Determine if the initial-value problem

$$y'(t) = \frac{2}{t}y + t^2e^t, \ 1 \leq t \leq 2, \ y(1) = 0$$

has a unique solution for $1 \leq t \leq 2$. If so, find the solution exactly or numerically.

Let $D = \{(t, y) | 1 \leq t \leq 2, \ -\infty < y < \infty\}$. Check if $f(t,y) = \frac{2}{t}y + t^2e^t$ satisfies a Lipschitz condition in $D$. Because

$$\left|\frac{\partial f}{\partial y}\right| = \left|\frac{2}{t}\right| \leq \frac{2}{1} = 2,$$

$f(t,y)$ satisfies a Lipschitz condition in $D$ and therefore by Theorem 5.4 the initial-value problem has a unique solution.

Solve $\frac{dy}{dt} = \frac{2}{t}y + t^2e^t$: 

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(i) Solve the homogeneous equation \( \frac{dy}{dt} - \frac{2}{t}y = 0 \) by separation of variables:

\[
\frac{1}{y} \frac{dy}{dt} = \frac{2}{t}, \quad \ln|y| = 2 \ln|t| + C, \quad e^{\ln|y|} = e^{\ln|t|^2 + C}, \quad y_h = t^2 e^C = t^2 C.
\]

(ii) Find a particular solution of the homogeneous equation \( \frac{dy}{dt} - \frac{2}{t}y = t^2 e^t \):

Let \( y_p = (At^2 + Bt + C)e^t \). Then

\[
y'_p = (2At + B)e^t + (At^2 + Bt + C)e^t
\]

and

\[
(2At + B)e^t + (At^2 + Bt + C)e^t - \frac{2}{t} (At^2 + Bt + C)e^t = t^2 e^t
\]

\[
A = 1, \quad B = 0, \quad C - B = 0, \quad C = 0
\]

The solution is \( y_p = t^2 e^t \) and the general solution is: \( y(t) = y_h + y_p = t^2 C + t^2 e^t \).

(iii) Solve \( C \) by the initial value \( y(1) = 0 \): \( y(1) = C + e = 0, \quad C = -e \).

The general solution: \( y(t) = t^2 e^t - et^2 \).

Example  Determine if the initial-value problem

\[
y'(t) = 1 + t \sin(yt), \quad 0 \leq t \leq 2, \quad y(0) = 0
\]

has a unique solution for \( 0 \leq t \leq 2 \). If so, find the solution exactly or numerically.

Let \( D = \{ (t, y) \mid 0 \leq t \leq 2, \quad -\infty < y < \infty \} \). Clearly, \( f(t, y) = 1 + t \sin(yt) \) is continuous on \( D \).

From an earlier example, we know \( f(t, y) = 1 + t \sin(yt) \) satisfies a Lipschitz condition on \( D \) in the variable \( y \). By Theorem 5.4, we know the initial value problem has a unique solution.

Solve the initial value problem: \( \frac{dy}{dt} = 1 + t \sin(yt), \quad y(0) = 0 \). We cannot solve it exactly. The numerical solution:

\[
y' = 1 + \sin(yt), \quad 0 \leq t \leq 2, \quad y(0) = 0
\]

5. Well-posed Initial-Value Problems:

Definition 5.5 The initial-value problem:

\[
y'(t) = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha
\]

is said to be a well-posed problem if

a. it has a unique solution \( y(t) \);
b. For any $\varepsilon > 0$, there exists a positive constant $k(\varepsilon)$, such that whenever $|\varepsilon_0| < \varepsilon$ and $\delta(t)$ is continuous with $|\delta(t)| < \varepsilon$ on $[a, b]$, the initial-value problem

$$z'(t) = f(t, z) + \delta(t), \ a \leq t \leq b, \ z(a) = \alpha + \varepsilon_0$$

has a unique solution $z(t)$ such that

$$|z(t) - y(t)| < \varepsilon \ k(\varepsilon).$$

Note that:

An initial-value problem is well-posed if a small change in the problem results only a small change in its solution. So, when small errors $\delta(t)$ and $\varepsilon_0$ are introduced in the differential equation and the initial value, respectively, the corresponding solution should be close to the solution of the original initial value problem.

**Example** Determine if the initial-value problem

$$y'(t) = y - t^2 + 1, \ 0 \leq t \leq 2, \ y(0) = 0.5$$

is a well-posed problem.

a. Solve the initial-value problem: $\frac{dy}{dt} = y - t^2 + 1$

(i) Solve the homogeneous equation $\frac{dy}{dt} - y = 0$ by separation of variables:

$$\frac{1}{y} \ dy = dt, \ \ln|y| = t + C, \ y_h = e^{t+C} = Ce^t.$$

(ii) Find a particular solution of the non-homogeneous equation: $\frac{dy}{dt} - y = -t^2 + 1$:

Let $y_p = At^2 + Bt + C$. Then $y'_p = 2At + B$, and $y''_p = 2A$.

$$2At + B - (At^2 + Bt + C) = -At^2 + (2A - B)t + (B - C) = -t^2 + 1.$$ So, $A = 1, \ 2A - B = 2 - B = 0, \ B = 2, \ B - C = 2 - C = 1, \ C = 1$. $y_p = t^2 + 2t + 1$.

(iii) Solve $C$ : $y(0) = 1 + C = 0.5, \ C = -0.5$

The general solution: $y(t) = y_h + y_p = -0.5e^t + t^2 + 2t + 1$.

b. Let $\delta$ and $\varepsilon_0$ be some small constants. Add the errors $\delta(t) = \delta t$ and $\varepsilon_0$ to the differential equations and the initial-value, respectively:

$$\frac{dz}{dt} = z - t^2 + 1 + \delta t, \ 0 \leq t \leq 2, \ z(0) = 0.5 + \varepsilon_0$$

(i)-(ii) The general solution of this initial value problem is:

$$z(t) = Ce^t - t\delta + t^2 + (2 - \delta)t + 1 - \delta.$$ Solve $C : z(0) = C + 1 - \delta = 0.5 + \varepsilon_0$, $C = -0.5 + \varepsilon_0 + \delta$ and then

$$z(t) = e^{t(-0.5 + \varepsilon_0 + \delta)} + t^2 + (2 - \delta)t + 1 - \delta.$$ Let $\varepsilon > 0, |\delta| < \varepsilon$ and $|\varepsilon_0| < \varepsilon$. Check $|z(t) - y(t)|$:

$$|z(t) - y(t)| = |e^{t(-0.5 + \varepsilon_0 + \delta)} + t^2 + (2 - \delta)t + 1 - \delta - (-0.5e^t + t^2 + 2t + 1)|$$

$$= |(\varepsilon_0 + \delta)e^t - \delta t - \delta| = |\delta(e^t - t - 1) + \varepsilon_0e^t| < |\delta(e^t - t - 1)| + |\varepsilon_0e^t|$$

$$< \varepsilon(e^2 - t - 1) + \varepsilon e^2 < \varepsilon(2e^2 - 1)$$

So, by the definition of well-posed initial-value problems, this initial-value problem is well-posed with $k(\varepsilon) = 2e^2 - 1$. 

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6. A Sufficient Condition for a Well-posed Problem:

**Theorem 5.6** Let $D = \{(t, y) : a \leq t \leq b, -\infty < y < \infty\}$. If $f$ is continuous and satisfies a Lipschitz condition in the variable $y$ on the set $D$, then the initial value problem

$$y'(t) = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha.$$ 

is well-posed.

**Example** Consider again the initial-value problem: $y'(t) = y - t^2 + 1$, $0 \leq t \leq 2$, $y(0) = 0.5$.

Let $D = \{(t, y) : 0 \leq t \leq 2, -\infty < y < \infty\}$. Let $f(t, y) = y - t^2 + 1$. Because $\frac{\partial f}{\partial y} = 1$, $f(t, y)$ satisfies a Lipschitz condition. Hence, by Theorem 5.6, the initial-value problem is well-posed.

7. Picard’s Method:

Picard’s method is a method approximate the solution $y(t)$ of the initial-value problem:

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

by a sequence of functions $\{y_k(t)\}$ where $y_k(t)$ are functions:

$$y_k(t) = \alpha + \int_a^t f(x, y_{k-1}(x))dx, \quad k = 1, 2, \ldots$$

That is, $y_0(t) = \alpha$,

$$y_1(t) = \alpha + \int_a^t f(x, \alpha)dx, \quad y_2(t) = \alpha + \int_a^t f(x, y_1(x))dx, \ldots$$

Derivation: For $k \geq 1$,

$$\int_a^t f(x, y(x))dx = \int_a^t y'(x)dx = y(x)|_a^t = y(t) - y(a) = y(t) - \alpha$$

implies

$$y_k(t) = \alpha + \int_a^t f(x, y_{k-1}(x))dx.$$ 

**Example** Find approximations $y_h(t)$ of the solution to the initial-value problem:

$$\frac{dy}{dt} = y + t, \quad y(0) = -1, \quad \text{by Picard’s method.}$$

For this problem, $f(t, y) = y + t$ and $y_0(t) = -1$. Then

$$y_1(t) = -1 + \int_0^t (-1 + x)dx = \frac{1}{2}t^2 - t - 1$$

$$y_2(t) = -1 + \int_0^t \left(\frac{1}{2}x^2 - x - 1 + x\right)dx = -1 + \int_0^t \left(\frac{1}{2}x^2 - 1\right)dx = \frac{1}{6}t^3 - t - 1$$

$$y_k(t) = \frac{1}{(k+1)!}t^{k+1} - t - 1.$$
Example  Find approximations $y_1(t)$ and $y_2(t)$ of the solution to the initial-value problem:
\[
\frac{dy}{dt} = 1 + t \sin(yt), \quad y(0) = 0, \text{ by Picard’s method.}
\]
For this problem, $f(t,y) = 1 + t \sin(yt)$ and $y_0(t) = 0$. Then
\[
y_1(t) = 0 + \int_0^t (1 + x \sin(x(0))) \, dx = \int_0^t dx = t.
\]
\[
y_2(t) = 0 + \int_0^t (1 + x \sin(x^2)) \, dx = t + t^2 \sin t^2.
\]

Example  Find approximations $y_h(t)$ of the solution to the initial-value problem:
\[
\frac{dy}{dt} = y^2 + t, \quad y(0) = 1, \text{ by Picard’s method.}
\]
\[
y_0(t) = 1, \quad y_1(t) = 1 + \int_0^t (1 + x) \, dx = t + \frac{1}{2} t^2 + 1
\]
\[
y_2(t) = 1 + \int_0^t \left( \left( x + \frac{1}{2} x^2 + 1 \right)^2 + x \right) \, dx = t + \frac{3}{2} t^2 + \frac{2}{3} t^3 + \frac{1}{4} t^4 + \frac{1}{20} t^5 + 1
\]
\[
y_3(t) = 1 + \int_0^t \left( \left( x + \frac{3}{2} x^2 + \frac{2}{3} x^3 + \frac{1}{4} x^4 + \frac{1}{20} x^5 + 1 \right)^2 + x \right) \, dx
\]
\[
= t + \frac{3}{2} t^2 + \frac{4}{3} t^3 + \frac{13}{12} t^4 + \frac{49}{60} t^5 + \frac{13}{30} t^6 + \frac{233}{1260} t^7 + \frac{29}{480} t^8 + \frac{31}{2160} t^9 + \frac{1}{400} t^{10} + \frac{1}{4400} t^{11} + 1
\]
black - $y_0$, blue - $y_1$, red - $y_2$, green - $y_3$