Gaussian Elimination - (3.1)

Consider solving a given system of n linear equations in n unknowns:

$$(*)\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}$$

where a_{ij} and b_i are constants and x_i are unknowns. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = [a_{ij}]_{i,j=1}^{n},\ b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = [b_i]_{i=1}^{n} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_j]_{j=1}^{n}.$$

Then the system (*) in matrix and vector notations is

$$Ax = b.$$

We know from linear algebra that the system $Ax = b$ may have a unique solution, infinitely many solutions or no solution. A system is said to be **consistent** if it has a solution. A system that has no solution is said to be **inconsistent**. The system has a unique solution if and only if the matrix A is nonsingular (invertible, i.e., A^{-1} exists) and the corresponding solution is: $x^* = A^{-1}b$. In this section, we study how to **determine systematically** if a given matrix A is invertible and how to find the solution when it is unique.

Idea: If A is an upper triangular matrix, then A is nonsingular and $Ax = b$ has a unique solution if $a_{ii} \neq 0$ for $i = 1,\ldots,n$. On the other hand, if $a_{ii} = 0$ for some i, then $Ax = b$ does not have a unique solution. How can we reduce A to an upper triangular matrix without the solution? We can use elementary row operations to reduce A to an upper triangular matrix U and the corresponding system $Ux = \bar{b}$ is e**quivalent** to the system $Ax = b$, that is, both systems have the same solution set.

Let $\bar{A} = \begin{bmatrix} A & b \end{bmatrix}$ be the augmented matrix of a given system of linear equations $Ax = b$.

1. **Elementary Row Operations**:

Let $R_1,\ R_2,\ldots,\ R_n$ be rows of A.

Row operations:

$$(a)\ cR_i \qquad (b)\ R_i + R_j \qquad (c)\ c_1R_i + c_2R_j \qquad (d)\ R_i \leftrightarrow R_j$$

Elementary Row Operations:

$$(1)\ R_i \leftrightarrow R_j \qquad (2)\ cR_i \to R_i \qquad (3)\ R_i + cR_j \to R_i$$

An elementary row operation is a row operation but a row operation may not be an elementary row operation mainly because of the difference in (2) and (c). An elementary row operation for $\begin{bmatrix} A & b \end{bmatrix}$ does not change the solution set of a linear system $Ax = b$. That is if $\begin{bmatrix} A & b \end{bmatrix} \xrightarrow{\text{elementary row operation}} \begin{bmatrix} B & \bar{b} \end{bmatrix}$ then $Ax = b$ and $Bx = \bar{b}$ have the same solution set.

Example Let $R_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, $R_2 = \begin{bmatrix} -4 & 5 & -6 \end{bmatrix}$.

a. $4R_1 + R_2 \to R_2 : R_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, $R_2 = \begin{bmatrix} 0 & 13 & 6 \end{bmatrix}$

b. $4R_1 + R_2 \to R_1 : R_1 = \begin{bmatrix} 0 & 13 & 6 \end{bmatrix}$, $R_2 = \begin{bmatrix} -4 & 5 & -6 \end{bmatrix}$

Computational complexity of elementary row operations: number of multiplications (1 division is counted as 1 multiplication) and number of additions required:

elementary row operation	number of multiplications	number of additions
$R_i \leftrightarrow R_j$	0	0
$cR_i \to R_i$	n	0
$R_i + cR_j \to R_i$	n	$n-1$

2. **Gaussian-elimination**:

Gaussian-elimination is a process of reducing a matrix to an upper triangular matrix for a square matrix or a generalized upper triangular matrix for a rectangular matrix by elementary row operations. For example:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \to \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \to \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & 0 & b_{33} & b_{34} \end{bmatrix}$$

The process should be done with using the minimum number of multiplications.

Elimination process:

Consider two rows: $\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$. Suppose that we want to eliminate the element a_{21}, that is we want the matrix to become the form of $\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \end{bmatrix}$. How many elementary row operations are needed and what is the number of multiplications required? In the case when $a_{11} \neq 0$, only one elementary row operation is needed:

$$\left(-\frac{a_{21}}{a_{11}}\right) R_1 + R_2 \to R_2$$

and the number of multiplications is 4 for

$$c = -\frac{a_{21}}{a_{11}},\ ca_{12},\ ca_{13},\ ca_{14}$$

Gaussian-elimination:

Start at the first column if $a_{11} \neq 0$ and eliminate all elements below a_{11} using elementary row operations

$$-\frac{a_{i1}}{a_{11}} R_1 + R_i \to R_i,\ i = 2, 3, \ldots n.$$

Continue the process for the 2nd column, the 3rd column 3, ..., the $(n-1)$th column. If for some $1 \leq i \leq n-1$, $a_{ii} = 0$, then find $a_{ji} \neq 0$ for $j > i$ and switch $R_i \leftrightarrow R_j$. If such a a_{ji} does not exist, then the process is broken down. If A is nonsingular, then the matrix can be triangularized without a break down (without considering the round off error.) What are the numbers of multiplications and additions needed?

Computational complexity of Gaussian elimination for an $n \times (n+1)$ **matrix**:

ith column	elementary row operation	multiplications	additions
1	$n-1$	$(n-1)(n+1)$	$(n-1)n$
2	$n-2$	$(n-2)n$	$(n-2)(n-1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	1	(1)3	(1)2

The total number of multiplications:

$$\sum_{i=1}^{n-1}(i)(i+2) = \sum_{i=1}^{n-1}(i^2+2i) = \frac{(n-1)n(2(n-1)+1)}{6} + 2\frac{(n-1)n}{2} = \frac{1}{2}n^2 - \frac{5}{6}n + \frac{1}{3}n^3$$

The total number of additions:

$$\sum_{i=1}^{n-1}(i)(i+1) = \sum_{i=1}^{n-1}(i^2+i) = \frac{(n-1)n(2(n-1)+1)}{6} + \frac{(n-1)n}{2} = \frac{1}{3}n^3 - \frac{1}{3}n$$

Hence, we have

the numbers of multiplications	$\frac{1}{2}n^2 - \frac{5}{6}n + \frac{1}{3}n^3$
the numbers of additions	$\frac{1}{3}n^3 - \frac{1}{3}n$

3. **Solving Systems of Linear Equations By Gaussian-elimination**:
Backward-substitution:

Let $U = [u_{ij}]_{i,j=1}^{n}$ be an upper triangular matrix, i.e., $u_{ij} = 0$ whenever $i > j$. Consider solving the system of linear equations:

$$Ux = b, \quad \text{where } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1(n-1)} & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2(n-1)} & u_{2n} \\ \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u_{(n-1)(n-1)} & u_{(n-1)n} \\ 0 & 0 & 0 & \cdots & 0 & u_{nn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}.$$

Note that the matrix U is nonsingular if and only if $u_{ii} \neq 0$ for $i = 1,\ldots,n$. Assume that U is nonsingular. Note that the nth equation: $u_{nn}x_n = b_n$ contains only one unknown x_n and it can be solved easily:

$$x_n = \frac{b_n}{u_{nn}}.$$

Now replace x_n by $\frac{b_n}{u_{nn}}$ in the first $n-1$ equations in the system and then the $(n-1)$th equations:

$$u_{(n-1)(n-1)}x_{n-1} + u_{(n-1)n}\left(\frac{b_n}{u_{nn}}\right) = b_{n-1}$$

contains only one unknown x_{n-1} and it can be solved easily:

$$x_{n-1} = \frac{1}{u_{(n-1)(n-1)}}\left(b_{n-1} - u_{(n-1)n}\left(\frac{b_n}{u_{nn}}\right)\right).$$

Continue the process to compute $x_{n-2},\ x_{n-3},\ldots,x_2,\ x_1$. Compute $x_n = \frac{b_n}{u_{nn}}$, and for $i = n-1, n-2,\ldots,2,1$,

$$x_i = \frac{1}{u_{ii}}(b_i - u_{i(i+1)}x_{i+1} - u_{i(i+2)}x_{i+2} - \ldots - u_{in}x_n).$$

What are the numbers of multiplications and additions needed? Let us count the following

i	multiplications	additions
n	1	0
$n-1$	2	1
$n-2$	3	2
$\vdots$	$\vdots$	$\vdots$
1	$n-1+1=n$	$n-1$

From the table at the left, we have

the numbers of multiplications	$1+2+\ldots+n=\dfrac{n(n+1)}{2}$
the numbers of additions	$1+2+\ldots+n-1=\dfrac{(n-1)n}{2}$

Gaussian elimination with backward substitution for solving $Ax=b$:

Step I Gaussian-elimination: $\begin{bmatrix} A & b \end{bmatrix} \to \begin{bmatrix} U & \bar{b} \end{bmatrix}$

The system does not have a unique solution if the elimination process breaks down.

Step II Backward-substitution: $u_{ii} \neq 0$ for $i=1,\ldots,n$.

Compute $x_n = \dfrac{\bar{b}_n}{u_{nn}}$, and for $i=n-1,n-2,\ldots,2,1$,

$$x_i = \frac{1}{u_{ii}}(\bar{b}_i - u_{i(i+1)}x_{i+1} - u_{i(i+2)}x_{i+2} - \ldots - u_{in}x_n).$$

Example *Let* $A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, *and* $\hat{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

Determine if $A_1x=b$, $A_1x=\hat{b}$ *and* $A_2x=b$ *have a unique solution, infinitely many solutions or no solution. Find the solution or the general solution if the system is consistent.*

(i) $A_1x=b$:

Step I $$\begin{bmatrix} A_1 & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{bmatrix} \xrightarrow[(-7)R_1+R_3\to R_3]{(-4)R_1+R_2\to R_2} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -3 \\ 0 & -6 & -12 & -6 \end{bmatrix} \xrightarrow{(-2)R_2+R_3\to R_3} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$A_1x=b$ is consistent and has infinitely many solutions.

Step II Solve the general solution: x_3 is free, let $x_3=t$, a real number

$$\begin{cases} x_2 = \dfrac{1}{(-3)}(-3-(-6)t) = 1-2t \\ x_1 = \dfrac{1}{1}(1-2(1-2t)-3(t)) = -1+t \end{cases}, \text{ the general solution: } x^* = \begin{bmatrix} -1+t \\ 1-2t \\ t \end{bmatrix}, \ t \text{ is a real number.}$$

MatLab:
```
>>A=[1 2 3;4 5 6;7 8 9]; b=[1;1;1]; B=[A b];
>>rref(B)
>> 1 0 -1 -1
   0 1 2  1
    0 0 0  0
```

(ii) $A_1x=\hat{b}$:

Step I $$\begin{bmatrix} A_1 & \hat{b} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & -1 \end{bmatrix} \xrightarrow[(-7)R_1+R_3\to R_3]{(-4)R_1+R_2\to R_2} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -3 \\ 0 & -6 & -12 & -8 \end{bmatrix} \xrightarrow{(-2)R_2+R_3\to R_3} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -3 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$A_1x=\hat{b}$ has no solution because the last equation: $0x_1+0x_2+0x_3=-2$ can never hold.

```
MatLab:  >>A=[1 2 3;4 5 6;7 8 9]; b=[1;1;-1]; B=[A b];
              >>rref(B)
              >> 1 0 -1 0
                 0 1  2 0
                  0 0  0 1
```

(iii) $A_2x = b$:

Step I $\begin{bmatrix} A_2 & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 10 & 1 \end{bmatrix} \underset{(-7)R_1+R_3 \to R_3}{\overset{(-4)R_1+R_2 \to R_2}{\longrightarrow}} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -3 \\ 0 & -6 & -11 & -6 \end{bmatrix} \overset{(-2)R_2+R_3 \to R_3}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$A_2x = b$ is consistent and has a unique solution.

Step II Solve the solution:

$$\begin{cases} x_3 = \dfrac{0}{1} = 0 \\ x_2 = \dfrac{1}{(-3)}(-3-(-6)(0)) = 1 \\ x_1 = \dfrac{1}{1}(1-2(1)-3(0)) = -1 \end{cases}, \text{ the solution: } x^* = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Check: $A_2x^* = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = b$

Use MatLab: `>>A=[1 2 3;4 5 6;7 8 10]; b=[1;1;1]; x=inv(A)*b`

Example *Find the inverse of* $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$ *if it exists using Gaussian Elimination and backward substitution.*

Let $C = A^{-1} = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix}$. We know that $AC = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Since

$$AC = \begin{bmatrix} AC_1 & AC_2 & AC_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

B can be obtained by solving systems: $AC_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $AC_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $AC_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Step I: Gaussian Elimination for $\begin{bmatrix} A & I_3 \end{bmatrix} \to \begin{bmatrix} U & B \end{bmatrix}$, $B = \begin{bmatrix} B_1 & B_2 & B_3 \end{bmatrix}$

$$\begin{bmatrix} A & I_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 10 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[(-7)R_1+R_3\to R_3]{(-4)R_1+R_2\to R_2} \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -11 & -7 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{(-2)R_2+R_3\to R_3} \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} U & B_1 & B_2 & B_3 \end{bmatrix}$$

Step II: Backward Substitution: solve $UC_1 = B_1,\ UC_2 = B_2,\ UC_3 = B_3$.

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -4 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{cases} x_3 = 1 \\ x_2 = \dfrac{1}{(-3)}(-4-(-6)(1)) = -\frac{2}{3} \\ x_1 = \dfrac{1}{1}\left(1-2\left(-\frac{2}{3}\right)-3(1)\right) = -\frac{2}{3} \end{cases} \text{, the solution: } B_1 = \begin{bmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \begin{cases} x_3 = -2 \\ x_2 = \dfrac{1}{(-3)}(1-(-6)(-2)) = \frac{11}{3} \\ x_1 = \dfrac{1}{1}\left(0-2\left(\frac{11}{3}\right)-3(-2)\right) = -\frac{4}{3} \end{cases} \text{, the solution: } B_2 = \begin{bmatrix} -\frac{4}{3} \\ \frac{11}{3} \\ -2 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{cases} x_3 = 1 \\ x_2 = \dfrac{1}{(-3)}(0-(-6)(1)) = -2 \\ x_1 = \dfrac{1}{1}(0-2(-2)-3(1)) = 1 \end{cases} \text{, the solution: } B_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

$$A^{-1} = B = \begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{11}{3} & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

Check: $AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{11}{3} & -2 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$