

## LU Decomposition (Factorization) - (3.5)

Let  $L$  be a lower triangular matrix with 1's on its diagonal and  $U$  be an upper triangular matrix. For a given  $n \times n$  matrix  $A$ , if a pair of  $L$  and  $U$  can be found such that  $A = LU$ , then  $L$  and  $U$  are called LU decomposition or factorization of  $A$ . Can we always find  $L$  and  $U$  for a given  $A$ ? If so, how?

### 1. Elementary matrices:

The matrix  $E$  is said to be an elementary matrix if  $B = EA$  where  $B$  is reduced from  $A$  by one elementary row operation.

Recall three Elementary Row Operations:

$$(1) R_i \leftrightarrow R_j \quad (2) cR_i \rightarrow R_i \quad (3) cR_i + R_j \rightarrow R_j$$

Let  $I$  be the  $n \times n$  identity matrix. Define  $E^{(1)}(i, j)$  as the matrix that is obtained from  $I$  by exchanging the  $i$ th row and  $j$ th row. Define  $E^{(2)}(i)$  as the matrix that is obtained from  $I$  by multiplying the  $i$ th row by  $c$ . Define  $E^{(3)}(i, j)$  as the matrix that is obtained from  $I$  by replacing the  $(j, i)$ th element by  $c$ . Then

$$B = EA$$

for each of three elementary row operations.

**Example** Let  $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$

$$(1) A \xrightarrow{R_1 \leftrightarrow R_3} B = \begin{bmatrix} 7 & 8 & 10 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}, \text{ let } E^{(1)}(1, 3) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \text{ then } E^{(1)}(1, 3)A = B.$$

$$(2) A \xrightarrow{2R_2 \rightarrow R_2} B = \begin{bmatrix} 1 & 2 & 3 \\ 8 & 10 & 12 \\ 7 & 8 & 10 \end{bmatrix}, \text{ let } E^{(2)}(2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ then } E^{(2)}(2)A = B.$$

$$(3) A \xrightarrow{-7R_1 + R_3 \rightarrow R_3} B_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & -6 & -11 \end{bmatrix} \xrightarrow{-4R_1 + R_3 \rightarrow R_3} B_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{bmatrix},$$

$$\text{let } E^{(3)}(1, 3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix} \text{ and let } E^{(3)}(1, 2) = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ then } E^{(3)}(1, 3)E^{(3)}(1, 2)A = B_2.$$

$$\text{Note that } E^{(3)}(1, 3)E^{(3)}(1, 2) = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}.$$

### 2. Gaussian Elimination with only Elementary Row Operations **Type III**:

Suppose  $A \xrightarrow{cR_i + R_j \rightarrow R_j} U$  by  $m$  elementary row operations type III where  $j > i$ . Let  $E(i, j)$  for  $i < j$  be the elementary matrix for the elementary row operation  $cR_i + R_j \rightarrow R_j$ . Then

$$E^{(3)}(n-1, n) \cdots E^{(3)}(1, 2)A = U,$$

and  $A = (E(n-1, n) \cdots E(1, 2))^{-1}U = E(1, 2)^{-1}E(1, 3)^{-1} \cdots E(n-1, n)^{-1}U$ . Note that

- a.  $E^{(3)}(i, j)^{-1}$  is again an **elementary matrix** for the elementary row operation  $(-c)R_i + R_j \rightarrow R_j$ . For example,

$$E^{(3)}(1, 2) = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E(1, 2)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- b.  $E^{(3)}(i, j)$ 's are lower triangular matrices with 1's on its diagonal and therefore,  $E^{(3)}(n-1, n) \cdots E^{(3)}(1, 2)$  and  $E^{(3)}(1, 2)^{-1}E^{(3)}(1, 3)^{-1} \cdots E^{(3)}(n-1, n)^{-1}$  are lower triangular matrices with 1's on its diagonal.

Let  $L = E^{(3)}(1, 2)^{-1}E^{(3)}(1, 3)^{-1} \cdots E^{(3)}(n-1, n)^{-1}$ . We have  $A = LU$ .

**Example** Let  $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$ . Find LU decomposition of A.

$$\text{Recall } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \xrightarrow[(-7)R_1+R_3 \rightarrow R_3]{(-4)R_1+R_2 \rightarrow R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{bmatrix} \xrightarrow{(-2)R_2+R_3 \rightarrow R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$E^{(3)}(1, 2) = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E^{(3)}(1, 3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix} \text{ and } E^{(3)}(2, 3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Then  $L = E^{(3)}(2, 3)^{-1}E^{(3)}(1, 3)^{-1}E^{(3)}(1, 2)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix}$ . Then  $A = LU$ . Check:

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}.$$

### 3. Gaussian Elimination with Elementary Row Operations **Type I and III**:

Let  $E^{(1)}(i, j)$  be the elementary matrix for the elementary row operation  $R_i \leftrightarrow R_j$ . Note that  $E^{(1)}(i, j)^{-1} = E^{(1)}(i, j)$ . Suppose that

$$E^{(3)}(n-1, n)E^{(1)}(n-1, n) \cdots E^{(3)}(1, 2)E^{(1)}(1, 2)A = U.$$

Then  $A = (E^{(3)}(n-1, n)E^{(1)}(n-1, n) \cdots E^{(3)}(1, 2)E^{(1)}(1, 2))^{-1}U$ . Let  $P = E^{(1)}(n-1, n) \cdots E^{(1)}(1, 2)$  and

$$L = P(E^{(3)}(n-1, n)E^{(1)}(n-1, n) \cdots E^{(3)}(1, 2)E^{(1)}(1, 2))^{-1}.$$

Then  $PA = LU$ .

**Example** Let  $A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 9 \end{bmatrix}$ . Find either LU factorization of  $A$  or  $PA$ .

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 9 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 6 & 7 & 9 \end{bmatrix} \xrightarrow{(-2)R_1 + R_3 \rightarrow R_3} \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow{R_2 + R_3 \rightarrow R_3} \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$E^{(1)}(1,2) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E^{(3)}(1,3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, E^{(3)}(2,3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$E^{(1)}(1,2)^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E^{(3)}(1,3)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, E^{(3)}(2,3)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\text{Let } P = E^{(1)}(1,2), \text{ and } L = E^{(1)}(1,2)E^{(1)}(1,2)E^{(3)}(1,3)^{-1}E^{(3)}(2,3)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 6 & 7 & 9 \end{bmatrix}$$

$$PA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 6 & 7 & 9 \end{bmatrix}$$

#### 4. Solving Systems $Ax = b$ when $A = LU$ or $PA = LU$ :

a.  $A = LU$ .

Then  $Ax = b$  can be solved by solving the following two systems:  $\begin{cases} Ly = b \\ Ux = y \end{cases}$ . Because  $L$  is a lower

triangular matrix,  $Ly = b$  can be solved by the forward substitution:

$$Ly = b \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}$$

$$\begin{aligned}
y_1 &= b_1 \\
y_2 &= b_2 - l_{21}y_1 \\
y_3 &= b_3 - l_{31}y_1 - l_{32}y_2 \\
&\vdots \\
y_n &= b_n - (l_{n1}y_1 + l_{n2}y_2 + \dots + l_{n(n-1)}y_{n-1}).
\end{aligned}$$

Then solve  $Ux = y$  for  $x$  by the backward substitution: assuming  $u_{ii} \neq 0$  for  $i = 1, \dots, n$

$$Ux = y \Leftrightarrow \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1(n-1)} & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2(n-1)} & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3(n-1)} & u_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u_{(n-1)(n-1)} & u_{(n-1)n} \\ 0 & 0 & 0 & \cdots & 0 & u_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}$$

$$\begin{aligned}
x_n &= \frac{y_n}{u_{nn}} \\
x_{n-1} &= \frac{1}{u_{(n-1)(n-1)}} (y_{n-1} - u_{(n-1)n}x_n) \\
x_{n-2} &= \frac{1}{u_{(n-1)(n-1)}} (y_{n-2} - u_{(n-2)(n-1)}x_{n-1} - u_{(n-2)n}x_n) \\
&\vdots \\
x_1 &= \frac{1}{u_{11}} (y_1 - u_{12}x_2 - u_{13}x_3 - \cdots - u_{1n}x_n).
\end{aligned}$$

What are the number of multiplications and additions for solving  $LUx = b$ ?

	$y$	$x$	total
$*, /$	$1 + 2 + \dots + n - 1 = \frac{(n-1)n}{2}$	$1 + 2 + \dots + n = \frac{n(n+1)}{2}$	$\frac{(n-1)n}{2} + \frac{n(n+1)}{2} = n^2$
$+, -$	$1 + 2 + \dots + n - 1 = \frac{(n-1)n}{2}$	$1 + 2 + \dots + n - 1 = \frac{(n-1)n}{2}$	$2 \frac{(n-1)n}{2} = n^2 - n$

b.  $PA = LU$  :

Observe that  $Ax = b \Leftrightarrow PAx = Pb = \bar{b}$ . So, we solve  $x$  by solving the system:  $LUx = \bar{b}$  in two steps as described in a. above.

## 5. LU factorization for symmetric matrices:

A square matrix  $A$  is symmetric if  $A^T = A$  (elements in the  $i$ th row are the same as the ones in the  $j$ th column). Let  $A$  be a symmetric matrix and  $A = LU$ . Since  $A^T = (LU)^T = U^T L^T = LU$ ,  $L$  and  $U$  should be related in some way. Let  $D$  be a diagonal matrix whose diagonal elements are diagonal elements of  $U$  :  $u_{11}, u_{22}, \dots, u_{nn}$ . Then

$$U = \begin{bmatrix} u_{11} & 0 & 0 & \cdots & 0 & 0 \\ 0 & u_{22} & 0 & \cdots & 0 & 0 \\ 0 & 0 & u_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & u_{(n-1)(n-1)} & 0 \\ 0 & 0 & 0 & \cdots & 0 & u_{nn} \end{bmatrix} \begin{bmatrix} 1 & \frac{u_{12}}{u_{11}} & \frac{u_{13}}{u_{11}} & \cdots & \frac{u_{1(n-1)}}{u_{11}} & \frac{u_{1n}}{u_{11}} \\ 0 & 1 & \frac{u_{23}}{u_{22}} & \cdots & \frac{u_{2(n-1)}}{u_{22}} & \frac{u_{2n}}{u_{22}} \\ 0 & 0 & 1 & \cdots & \frac{u_{3(n-1)}}{u_{33}} & \frac{u_{3n}}{u_{33}} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{u_{(n-1)n}}{u_{(n-1)(n-1)}} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$= D \bar{U}$$

Now  $A = LD\bar{U} = A^T = \bar{U}^TDL^T$  so  $\bar{U}^T = L$  or  $\bar{U} = L^T$ . When  $A$  symmetric, then  $A = LDL^T$  or  $PA = LDL^T$ .

Steps to solve  $Ax = b$  when  $A = LDL^T$  :

$$Ax = LDL^T x = b \Leftrightarrow L \underbrace{(DL^T x)}_y = b$$

$$DL^T x = y \Leftrightarrow L^T x = D^{-1}y = \bar{y}$$

- Solve  $Ly = b$  for  $y$  by a forward substitution.
- Compute  $\bar{y} = D^{-1}y$ .
- Solve  $L^T x = \bar{y}$  for  $x$  by a backward substitution.

**Example** Let  $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ . (1) Find  $L$  and  $D$  such that  $A = LDL^T$  or  $PA = LDL^T$ . (2) Solve

$$Ax = b, \text{ where } b = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

$$(1) A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \xrightarrow[(-3)R_1+R_3 \rightarrow R_3]{(-2)R_1+R_2 \rightarrow R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & -3 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{bmatrix} = U$$

$$E^{(1)}(2,3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E^{(3)}(1,2) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E^{(3)}(1,3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E^{(1)}(2,3)E^{(3)}(1,2)E^{(3)}(1,3)A = U$$

$$L = E^{(1)}(2,3)(E^{(1)}(2,3)E^{(3)}(1,3)E^{(3)}(1,2))^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Let  $P = E^{(1)}(2,3)$ . Then

$$PA = LU =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = LDL^T$$

(2) Solve  $Ax = b$  :  $PAx = Pb = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \bar{b}$ . Now we solve  $LDL^T x = \bar{b}$ .

a. Solve  $Ly = \bar{b}$  for  $y$  :

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{array}{l} y_1 = 1 \\ y_2 = 1 - 3(1) = -2 \\ y_3 = -1 - 2(1) = -3 \end{array} \quad \therefore y = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$$

b. Compute  $\bar{y} = D^{-1}y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

c. Solve  $L^T x = \bar{y}$  for  $x$  by a backward substitution.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{array}{l} x_3 = 3 \\ x_2 = 2 - 3(3) = -7 \\ x_1 = 1 - (2)(-7) - 3(3) = x_1 = 6 \end{array} \quad \therefore x = \begin{bmatrix} 6 \\ -7 \\ 3 \end{bmatrix}$$

Check:  $Ax = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} 6 \\ -7 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = b$ .