Inverse Power Method, Shifted Power Method and Deflation - (4.2)(4.3)

Let A be an $n \times n$ real matrix and (λ_i, v_i) for $i = 1, \ldots, n$ be eigenpairs of A where

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|.$$

When $|\lambda_1| > |\lambda_2|$, the Power Method approximates (λ_1, v_1). How can other eigenvalues and their corresponding eigenvectors be approximated?

1. **Inverse Power Method -**

Property: Let A be nonsingular. If (λ_i, v_i) is an eigenpair of A, then $\left(\frac{1}{\lambda_i}, v_i\right)$ is an eigenpair of A^{-1}.

Proof: Since $Av_i = \lambda_i v_i$, $\frac{1}{\lambda_i} v_i = A^{-1} v_i$. So, $\left(\frac{1}{\lambda_i}, v_i\right)$ is an eigenpair of A^{-1}.

Clearly, $|\lambda_{n-1}| > |\lambda_n|$ implies $\frac{1}{|\lambda_n|} > \frac{1}{|\lambda_{n-1}|}$. Since $\frac{1}{|\lambda_n|} > \frac{1}{|\lambda_{n-1}|} \geq \cdots \geq \frac{1}{|\lambda_1|}$, the Power Method finds $\frac{1}{|\lambda_n|}$ for A^{-1} or $|\lambda_n|$ for A. Hence, (λ_n, v_n) can be computed as follows: Let $A = LU$.

Algorithm: Given A, x and a stopping criterion ε, let $x^{(0)} = \frac{1}{\|x\|_\infty} x$. Because $\|x^{(0)}\|_\infty = 1$, let $|x_{p_0}| = \|x^{(0)}\|_\infty = 1$.

For $k = 1, 2, \ldots,$

(1) Solve $y^{(k)}$ from the equation: $LUy^{(k)} = x^{(k-1)}$ (instead of $y^{(k)} = Ax^{(k-1)}$).

(2) Compute p_k such that $\left|y^{(k)}_{p_k}\right| = \|y^{(k)}\|_\infty$.

(3) Let $r_k = y^{(k)}_{p_k}$ and $x^{(k)} = \frac{1}{r_k} y^{(k)}$ ($\|x^{(k)}\|_\infty = 1$)

(4) If $\|x^{(k)} - x^{(k-1)}\|_\infty < \varepsilon$, then $\lambda_n \approx \frac{1}{r_k}$ and $v_n = x^{(k)}$. Otherwise, $k = k + 1$, repeat steps (1)-(4).

Example $A = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix}$, $x^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Eigenvalues of A are: $2, 3, 6$.

```
>>[xsol,rv,flag,k]=PowerMethod(inv(A),ones(3,1),3,10^(-8),100);
>>1/rv(k+1)
2.00000002822991
```

2. **Shifted Power Method**:

Property: Let $B = A - \lambda_1 I$. Then $(0, v_1)$, $(\lambda_i - \lambda, v_i)$ for $i = 2, \ldots, n$ are eigenpairs of B.

Proof: $Bv_1 = Av_1 - \lambda_1 v_1 = \lambda_1 v_1 - \lambda_1 v_1 = 0v_1$, and $Bv_i = Av_i - \lambda_1 v_i = (\lambda_i - \lambda_1)v_i$.

   a. The largest eigenvalue (in module) of B gives the eigenvalue $\lambda_i$of A that is the furthest away from λ_1.
   b. Let $q \neq \lambda_i$ for $i = 1, \ldots, n$ and $B = (A - qI)^{-1}$. $\dfrac{1}{\text{the largest eigenvalue (in module)of } B}$ gives the eigenvalue of A that is nearest to q. Note that if we know there is only one single eigenvalue in a Gerschgorin circle with center q, then this eigenvalue can be approximated by the Power Method with $B = A - qI$.

Example $A = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix}$, $x^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Eigenvalues of A are: $2, 3, 6$.

```
>> [xsol,rv,flag,k]=PowerMethod(A-6*eye(3),ones(3,1),3,10^(-8),100);
>> rv(k+1)+6
2.00000004036276
>> [xsol,rv,flag,k]=PowerMethod(A-eye(3),ones(3,1),3,10^(-8),100);
>> rv(k+1)+1
6.00000010217939
>> [xsol,rv,flag,k]=PowerMethod(inv(A-eye(3)),ones(3,1),3,10^(-8),100)
>>1/rv(k+1)+1
2.0000
```

Example $A = \begin{bmatrix} 16 & -8 & 2 & 1 \\ 2 & -12 & 1 & 0 \\ -1 & 1 & -4 & 1 \\ 0 & -1 & 2 & 3 \end{bmatrix}$. *(1) Find regions which containe all eigenvalues of A. (2) Use the Power Method to approximate as many eigenvalues of A as possible.*

(1)

(2)

3. Deflation:

Property: Matrices A and A^T have the same set of eigenvalues.

Property: Let (λ_i, w_i) be eigenpairs of A^T, i.e., $A^T w_i = \lambda_i w_i$. If $\lambda_i \neq \lambda_j$, then $w_i^T v_j = 0$.

Proof: $0 = w_i^T (A v_j) - v_j^T A^T w_i = \lambda_j (w_i^T v_j) - \lambda_i (v_j^T w_i) = (\lambda_j - \lambda_i) w_i^T v_j \Leftrightarrow w_i^T v_j = 0$.

Property: Let $B = A - \lambda_i v_i x^T$, where $v_i^T x = 1$. Then eigenvalues of B are 0, and λ_j for $j \neq i$.

Proof:

$$B v_i = A v_i - \lambda_i v_i x^T v_i = (\lambda_i - \lambda_i(1)) v_i = 0.$$

For $j \neq i$,

$$B^T w_j = (A - \lambda_i v_i x^T)^T w_j = A^T w_j - \lambda_i x (v_i^T w_j) = A^T w_j = \lambda_j w_j.$$

In the case where $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$, we can use the Power Method to find $\lambda_1, \lambda_2, \ldots, \lambda_n$ one by one by deflation using matrices: $B_1 = A$, $B_2 = B_1 - \lambda_1 u_1 x_1^T$, $B_3 = B_2 - \lambda_2 u_2 x_2^T$, ...

Question: How to construct x_i?

Two methods:

a. Hotelling Deflation Method:

$$x_i = \frac{1}{\|u_i\|_2^2} u_i, \; u_1 = v_1$$

b. Wielandt Deflation Method:

$x_i = \frac{1}{\lambda_i u_{ii}} B_{ii}^T$ where u_{ii} is the $i$the element of u_i and B_{ii} is the ith row of B_i, $u_1 = v_1$.

Example $A = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix}$, $x^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

By Hotelling Deflation Method:

```
>>B1=A;
>> [xsol,rv,flag,k]=PowerMethod(B1,ones(3,1),3,10^(-8),100);
>> r1=rv(k+1);
>> u1=xsol;
>> x1=u1/norm(u1,2)^2;
>> B2=B1-r1*u1*x1';
>> [xsol,rv,flag,k]=PowerMethod(B2,ones(3,1),3,10^(-8),100);
>> r2=rv(k+1);
>> u2=xsol;
>> x2=u2/norm(u2,2)^2;
>> B3=B2-r2*u2*x2';
>> [xsol,rv,flag,k]=PowerMethod(B3,ones(3,1),3,10^(-8),100);
>> r3=rv(k+1);
>> u3=xsol;
>> [r1 r2 r3]
6.0000 3.0000 2.0000
```

By Weilandt Deflation Method:

```
>>B1=A;
>>[xsol,rv,flag,k]=PowerMethod(B1,ones(3,1),3,10^(-8),100);
>>r1=rv(k+1);
>>u1=xsol;
>>x1=B1(1,1:3)'/(r1*u1(1));
>>B2=B1-r1*u1*x1';
>>[xsol,rv,flag,k]=PowerMethod(B2,ones(3,1),3,10^(-8),100);
>>r2=rv(k+1);
>>u2=xsol;
>>x2=B2(2,1:3)'/(r2*u2(2));
>>B3=B2-r2*u2*x2';
>>[xsol,rv,flag,k]=PowerMethod(B3,ones(3,1),3,10^(-8),100);
>>r3=rv(k+1);
>>u3=xsol;
>>[r1 r2 r3]
6.0000 3.0000 2.0000
```

Next question: How to form eigenvectors of A?

Let $B = A - \lambda_1 v_1 x^T$ where $v_1^T x = 1$ and $(0, u_1)$ and (λ_i, u_i) for $i = 2, \ldots, n$ be eigenpairs of B. We know $u_1 = v_1$. Let $v_i = a\, u_i + b\, v_1$. What are values of a and b? Observe that

$$Bv_i = (A - \lambda_1 v_1 x^T)v_i = Av_i - \lambda_1(x^T v_i)\, v_1 = \lambda_i v_i - \lambda_1(x^T v_i)\, v_1.$$

Left:

$$Bv_i = B\big(a\, u_i + b\, v_1\big) = aBu_i + bBv_1 = a\lambda_i u_i + b(0)v_1 = a\lambda_i u_i;$$

right:

$$\begin{aligned}\lambda_i v_i - \lambda_1(x^T v_i)\, v_1 &= \lambda_i(au_i + bv_1) - \lambda_1(x^T(au_i + bv_1))\, v_1\\ &= a\lambda_i u_i + b\lambda_i v_1 - \lambda_1 a(x^T u_i)v_1 - \lambda_1 b(x^T v_1)v_1\\ &= a\lambda_i u_i + b\lambda_i v_1 - \lambda_1 a(x^T u_i)v_1 - \lambda_1 b(1)v_1.\end{aligned}$$

$$Bv_i = B\big(a\, u_i + b\, v_1\big) \Leftrightarrow a\lambda_i u_i = a\lambda_i u_i + b\lambda_i v_1 - \lambda_1 a(x^T u_i)v_1 - \lambda_1 b(1)v_1$$

or

$$b\lambda_i v_1 - \lambda_1 a(x^T v_1)v_1 - \lambda_1 bv_1 = 0 \text{ or } (b(\lambda_i - \lambda_1) - \lambda_1 a(x^T u_i))v_1 = 0.$$

One solution to the equation is to let $a = \lambda_i - \lambda_1$ and $b = \lambda_1(x^T v_1)$ and

$$v_i = (\lambda_i - \lambda_1)\, u_i + \lambda_1(x^T u_i)\, v_1.$$

Example *Previous example. Compute also eigenvectors of A.*

```
[xsol,rv,flag,k]=PowerMethod(B1,ones(3,1),3,10^(-8),100);
r1=rv(k+1);
u1=xsol;
x1=B1(1,1:3)’/(r1*u1(1));
v1=u1;
B2=B1-r1*u1*x1’;
[xsol,rv,flag,k]=PowerMethod(B2,ones(3,1),3,10^(-8),100);
r2=rv(k+1);
u2=xsol;
x2=B2(2,1:3)’/(r2*u2(2));
v2=(r2-r1)*u2+r1*(x1’*u2)*u1;
B3=B2-r2*u2*x2’;
[xsol,rv,flag,k]=PowerMethod(B3,ones(3,1),3,10^(-8),100);
r3=rv(k+1);
u3=xsol;
v3=(r3-r2)*u3+r2*(x2’*u3)*u2;
[r1 r2 r3]
[A*v1-r1*v1 A*v2-r2*v2 A*v3-r3*v3]
1.0e-006 *
-0.0255 -0.1022 -0.0000
-0.0237 -0.1603 -0.0000
-0.0128 -0.1274 0.0000
```