The Elementary Theory of Initial–Value Problems - (7.1)

In Chapter 7, we study the numerical methods for solving initial-value problems for ordinary differential equations:

$$y'(t) = f\big(t,\ y\big),\ a \le t \le b,\quad y(a) = \alpha.$$

1. Lipschitz Condition:

Definition 1 *A function $f(t,y)$ is said to satisfy a* ***Lipschitz condition*** *in the variable y on a set $D \subset R^2$ if there exists a constant $L > 0$ such that*

$$|\, f(t,y_1) - f(t,y_2)\,| \le L\,|\, y_1 - y_2\,|,$$

whenever both points $(t,\ y_1)$ and $(t,\ y_2)$ are in D. The constant L is called a ***Lipschitz constant*** *for f.*

Example *Let $f(t,y) = 1 + ty^2$ and $D = \{\,(t,y) \mid 0 \le t \le 2,\ \ -1 \le y \le 1\}$. Does f satisfy a Lipschitz condition on D? If so, find its Lipschitz constant.*

Let $(t,\ y_1)$ and $(t,\ y_2)$ be in D, i.e., t is in $[0,2]$ and y is in $[-1,\ 1]$. Observe that

$$|\, f(t,y_1) - f(t,y_2)\,| = |1 + ty_1^2 - (1 + ty_2^2)| = |\,t\,||y_1^2 - y_2^2| = |\,t\,||(y_1 + y_2)(y_1 - y_2)| = |\,t\,||\,y_1 + y_2\,||\,y_1 - y_2\,|.$$

Because $|\,t\,| \le 2$ and $|\,y_1 + y_2\,| \le 2$, we have

$$|\, f(t,y_1) - f(t,y_2)\,| \le (2)(2)|y_1 - y_2| = 4|\,y_1 - y_2\,|.$$

So, f satisfies a Lipschitz condition and its Lipschitz constant is 4.

Note that the Lipschitz constant L is not unique, that is, for any $\bar{L} > 4$, the inequality

$$|\, f(t,y_1) - f(t,y_2)\,| \le \bar{L}\,|\,y_1 - y_2\,|$$

also holds. So, in practice, we want to find a L as small as possible.

2. A Sufficient Condition for Lipschitz Condition:

Theorem 1 *Suppose $f(t,y)$ is defined on a convex set $D \subset R^2$. If a constant $L > 0$ exists with*

$$\left|\frac{\partial f}{\partial y}\big(t,\ y\big)\right| \le L,\ \text{for all}\ \big(t,\ y\big) \in D,$$

then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L.

Example *Let $f(t,y) = 1 + ty^2$ and $D = \{\,(t,y) \mid 0 \le t \le 2,\ \ -1 \le y \le 1\}$. Does f satisfy a Lipschitz condition on D? If so, find its Lipschitz constant.*

The partial derivative of f with respect to y is $\frac{\partial f}{\partial y}\big(t,\ y\big) = 2ty$. Because

$$\left|\frac{\partial f}{\partial y}\big(t,\ y\big)\right| = |\,2ty\,| = 2|\,t\,||\,y\,| \le 2(2)(1) = 4,$$

f satisfies a Lipschitz condition with Lipschitz constant 4.

Example *Let $f(t,y) = 1 + t\sin(ty)$. Determine if f satisfies a Lipschitz condition in $D = \{\big(t,\ y\big) \mid 0 \le t \le 2,\ -\infty < y < \infty\}$.*

Compute the partial derivative of f with respect to y : $\frac{\partial f}{\partial y}(t, y) = t\cos(ty)\,(t) = t^2\cos(ty)$. Because $|\cos(ty)| \le 1$ and $|t| \le 2$,

$$\left|\frac{\partial f}{\partial y}(t, y)\right| = |t^2\cos(ty)| = |t^2|\,|\cos(ty)| \le |t^2| \le 4.$$

So, f satisfies a Lipschitz condition with a constant 4.

3. A Sufficient Condition for the Uniqueness of Solution of an Initial Value Problem:

Theorem *2 Suppose that* $D = \{(t, y) \mid a \le t \le b, \ -\infty < y < \infty\}$ *and that* $f(t, y)$ *is continuous on* D. *If* f *satisfies a Lipschitz condition on* D *in the variable* y, *then the initial-value problem*

$$y'(t) = f(t, y), \ a \le t \le b, \ y(a) = \alpha$$

has a ***unique solution*** $y(t)$ *for* $a \le t \le b$.

Example *Determine if the initial - value problem*

$$y'(t) = \frac{2}{t}y + t^2e^t, \ \ 1 \le t \le 2, \ y(1) = 0$$

has a unique solution for $1 \le t \le 2$. *If so, find the solution exactly or numerically.*

Let $D = \{(t, y) \mid 1 \le t \le 2, \ -\infty < y < \infty\}$. Check if $f(t, y) = \frac{2}{t}y + t^2e^t$ satisfies a Lipschitz condition in D. Because

$$\left|\frac{\partial f}{\partial y}\right| = \left|\frac{2}{t}\right| \le \frac{2}{1} = 2,$$

$f(t, y)$ satisfies a Lipschitz condition in D and therefore by Theorem 2 the initial-value problem has a unique solution.

Solve $\frac{dy}{dt} = \frac{2}{t}y + t^2e^t$:

(i) Solve the homogeneous equation $\frac{dy}{dt} - \frac{2}{t}y = 0$ by separation of variables:

$$\frac{1}{y}dy = \frac{2}{t}dt, \quad \ln|y| = 2\ln|t| + C, \quad e^{\ln|y|} = e^{\ln|t|^2 + C}, \ y_h = t^2e^C = t^2C.$$

(ii) Find a particular solution of the homogeneous equation $\frac{dy}{dt} - \frac{2}{t}y = t^2e^t$:

Let $y_p = (At^2 + Bt + C)e^t$. Then

$$y_p' = (2At + B)e^t + (At^2 + Bt + C)e^t$$

and

$$(2At + B)e^t + (At^2 + Bt + C)e^t - \frac{2}{t}(At^2 + Bt + C)e^t = t^2e^t \ \Leftrightarrow$$

$$At^2e^t + (2A + B - 2A)te^t + (B + C - 2B)e^t - \frac{2}{t}Ce^t = t^2e^t$$

$$A = 1, \ B = 0, \ C - B = 0, \ C = 0$$

The solution is $y_p = t^2e^t$ and the general solution is: $y(t) = y_h + y_p = t^2C + t^2e^t$.

(iii) Solve C by the initial value $y(1) = 0$: $y(1) = C + e = 0$, $C = -e$.

The general solution: $y(t) = t^2e^t - et^2$.

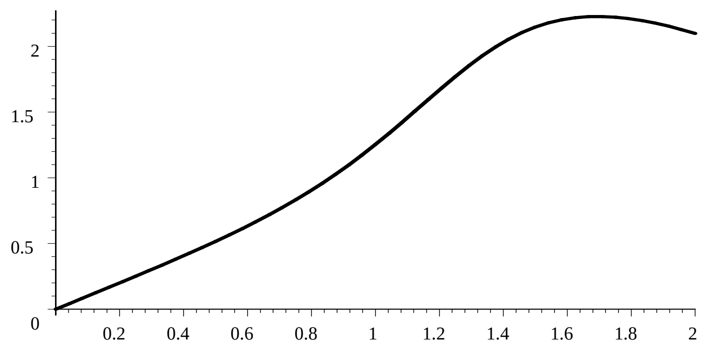
Example *Determine if the initial - value problem*

$$y'(t) = 1 + t\sin(yt),\ 0 \le t \le 2,\ y(0) = 0$$

has a unique solution for $0 \le t \le 2$. *If so, find the solution exactly or numerically.*

Let $D = \{(t,\ y)\ |\ 0 \le t \le 2,\ \ -\infty < y < \infty\}$. Clearly, $f(t,\ y) = 1 + t\sin(yt)$ is continuous on D. From an earlier example, we know $f(t,\ y) = 1 + t\sin(yt)$ satisfies a Lipschitz condition on D in the variable y. By Theorem 2, we know the initial value problem has a unique solution.

Solve the initial value problem: $\dfrac{dy}{dt} = 1 + t\sin(yt),\ y(0) = 0$. We cannot solve it exactly. The numerical solution:



$$y' = 1 + \sin(yt),\ 0 \le t \le 2,\ y(0) = 0$$

4. **Picard's Method**:

Picard's method is a method approximate the solution $y(t)$ of the initial-value problem:

$$y' = f(t,\ y),\ a \le t \le b,\ y(a) = \alpha$$

by a sequence of functions $\{\ y_k(t)\ \}$ where $y_k(t)$ are functions:

$$y_k(t) = \alpha + \int_a^t f(x,\ y_{k-1}(x))dx,\ k = 1, 2, \ldots.$$

That is, $y_0(t) = \alpha$,

$$y_1(t) = \alpha + \int_a^t f(x,\ \alpha)dx, \qquad y_2(t) = \alpha + \int_a^t f(x,\ y_1(x))dx,\ \ldots.$$

Derivation: For $k \ge 1$,

$$\int_a^t f(x, y(x))dx = \int_a^t y'(x)dx = y(x)|_a^t = y(t) - y(a) = y(t) - \alpha$$

implies

$$y_k(t) = \alpha + \int_a^t f(x,\ y_{k-1}(x))dx.$$

Example *Find approximations* $y_k(t)$ *of the solution to the initial-value problem:*

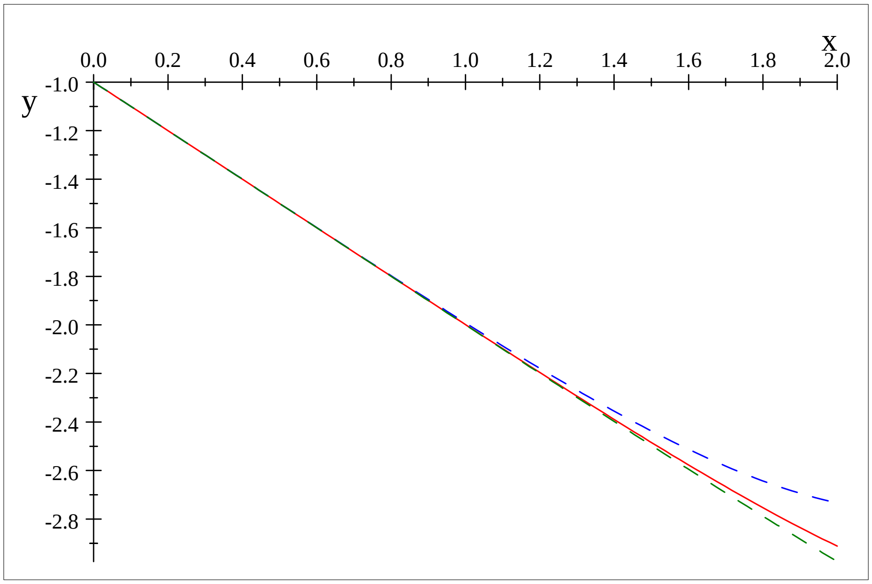
$\dfrac{dy}{dt} = y + t,\ y(0) = -1,$ *by Picard's method.*

For this problem, $f(t, y) = y + t$ and $y_0(t) = -1$. Then

$$y_1(t) = -1 + \int_0^t (-1 + x)dx = \frac{1}{2}t^2 - t - 1$$

$$y_2(t) = -1 + \int_0^t \left(\frac{1}{2}x^2 - x - 1 + x\right)dx = -1 + \int_0^t \left(\frac{1}{2}x^2 - 1\right)dx = \frac{1}{6}t^3 - t - 1$$

$$y_k(t) = \frac{1}{(k+1)!}t^{k+1} - t - 1.$$



.-.- $y_4(t)$, $-\, y_5(t)$, - - $y_6(t)$

Example *Find approximations* $y_1(t)$ *and* $y_2(t)$ *of the solution to the initial-value problem:* $\frac{dy}{dt} = 1 + t\sin(yt)$, $y(0) = 0$, *by Picard's method.*

For this problem, $f(t,y) = 1 + t\sin(yt)$ and $y_0(t) = 0$. Then

$$y_1(t) = 0 + \int_0^t (1 + x\sin(x(0)))dx = \int_0^t dx = t.$$

$$y_2(t) = 0 + \int_0^t (1 + x\sin(x^2))dx = t + t^2\sin t^2.$$

$$y_3(t) = 0 + \int_0^t (1 + x\sin(x(x + x^2\sin(x^2))))dx = ?$$

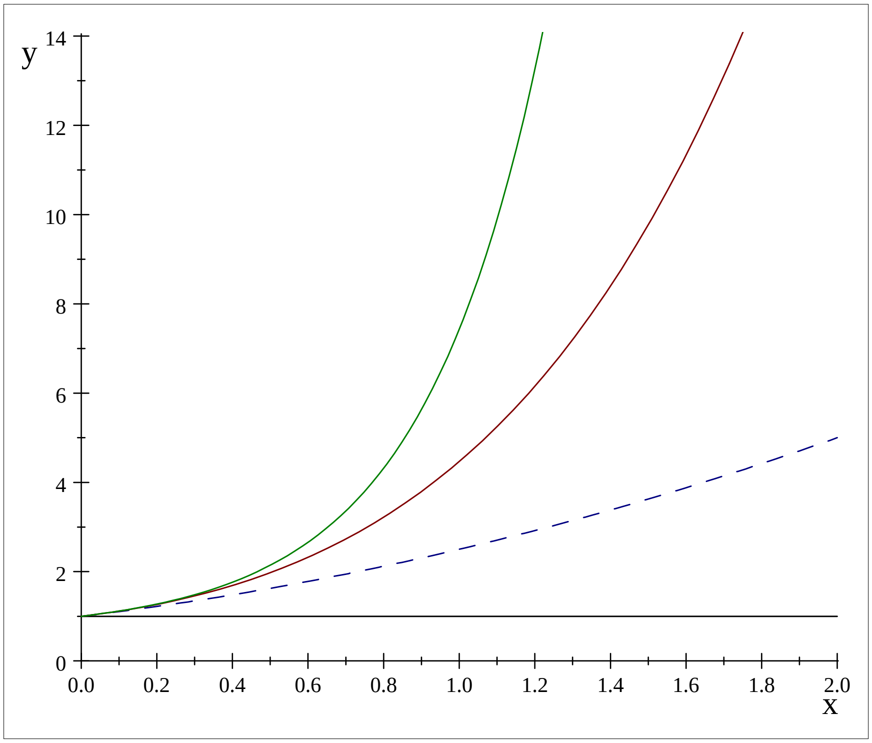
Example *Find approximations* $y_3(t)$ *of the solution to the initial-value problem:* $\frac{dy}{dt} = y^2 + t$, $y(0) = 1$, *by Picard's method.*

$$y_0(t) = 1,\ \ y_1(t) = 1 + \int_0^t (1 + x)dx = t + \frac{1}{2}t^2 + 1$$

$$y_2(t) = 1 + \int_0^t \left(\left(x + \frac{1}{2}x^2 + 1\right)^2 + x\right)dx = t + \frac{3}{2}t^2 + \frac{2}{3}t^3 + \frac{1}{4}t^4 + \frac{1}{20}t^5 + 1$$

$$y_3(t) = 1 + \int_0^t \left(\left(x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{20}x^5 + 1\right)^2 + x\right)dx$$

$$= t + \frac{3}{2}t^2 + \frac{4}{3}t^3 + \frac{13}{12}t^4 + \frac{49}{60}t^5 + \frac{13}{30}t^6 + \frac{233}{1260}t^7 + \frac{29}{480}t^8 + \frac{31}{2160}t^9 + \frac{1}{400}t^{10} + \frac{1}{4400}t^{11} + 1$$



black - y_0, blue - y_1, red - y_2, green - y_3

5. Approximate the solution by Taylor series:

Example *Determine the first 5 terms in Taylor series expansion of the solution to the initial-value problem:* $\frac{dy}{dt} = y + t,\ y(0) = -1.$

Let $y(t) = y(0) + y'(0)\,t + \frac{1}{2}y''(0)\,t^2 + \ldots + \frac{1}{n!}y^{(n)}(t)\,t^n + \ldots.$ and $y'(t) = f(t,y) = y + t.$

$y(0) = -1,\ y'(0) = y(0) + 0 = -1$

$y''(t) = y'(t) + 1,\ y''(0) = y'(0) + 1 = -1 + 1 = 0$

$y'''(t) = y''(t),\ y'''(0) = 0$

$y(t) \approx -1 - t$

Example *Determine the first 5 terms in Taylor series expansion of the solution to the initial-value problem:* $y' = t^2 - 2y^2 - 1,\ y(0) = 0.$

Let $y(t) = y(0) + y'(0)\,t + \frac{1}{2}y''(0)\,t^2 + \ldots + \frac{1}{n!}y^{(n)}(t)\,t^n + \ldots.$ and $f(t,y) = t^2 - 2y^2 - 1.$

$y(0) = 0,\ y'(0) = -1$

$y''(t) = 2t - 2(2yy') = 2t - 4yy',\ y''(0) = 0 - 4(9) = 0$

$y'''(t) = 2 - 4(y')^2 - 4yy'' = 2 - 4(-1)^2 = -2$

$y^{(4)}(t) = -8y'y'' - 4(y'y'' + yy'''),\ \ y^{(4)}(0) = 0 - 4(0 + 0) = 0$

$y(t) \approx -t - 2t^3$