Higher-Order Taylor Method - (7.3)

Consider the initial-value problem for ordinary differential equation:
\[ y'(t) = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha. \]
Suppose that the initial-value problem is well-posed and has a unique solution \( y(t) \). Let \( h = \frac{b-a}{N} \) for a given positive integer \( N \) and let \( t_i = t_{i-1} + h \) or \( t_i = a + ih \) for \( i = 0, \ldots, N \).

1. **Taylor Method of order \( n \):**
   Recall that Euler’s Method is derived using Taylor Method of order 1. Now we derive a difference method using Taylor Method of order \( n \). We first express \( y(t_{i+1}) \) by its \( n \)th Taylor polynomial and corresponding remainder:
   \[
   y(t_{i+1}) = y(t_i) + y'(t_i)h + \frac{y''(t_i)}{2!}h^2 + \ldots + \frac{y^{(n)}(t_i)}{n!}h^n + \frac{y^{(n+1)}(t^*_i)}{(n+1)!}h^{n+1}, \quad \text{where} \quad t_i \leq t^*_i \leq t_{i+1}
   \]
   
   \[
   = y(t_i) + f(t_i, y(t_i))h + \frac{f'(t_i, y(t_i))}{2!}h^2 + \ldots + \frac{f^{(n-1)}(t_i, y(t_i))}{n!}h^n + \frac{f^{(n)}(t^*_i, y(t^*_i))}{(n+1)!}h^{n+1}
   \]
   
   \[
   = y(t_i) + h \left( f(t_i, y(t_i)) + \frac{f'(t_i, y(t_i))}{2!}h + \ldots + \frac{f^{(n-1)}(t_i, y(t_i))}{n!}h^{n-1} + \frac{f^{(n)}(t^*_i, y(t^*_i))}{(n+1)!}h^n \right)
   \]

   Let
   \[
   T_n(t, y(t)) = f(t, y(t)) + \frac{f'(t, y(t))}{2!}h + \ldots + \frac{f^{(n-1)}(t, y(t))}{n!}h^{n-1}
   \]
   **Taylor Method of order \( n \):**
   
   \[
   y_0 = \alpha
   \]
   \[
   y_{i+1} = y_i + hT_n(t_i , y_i), \quad i = 0, 1, 2, \ldots, N - 1,
   \]

   Note that \( T_1(t, y(t)) = f(t, y(t)) \). So, Euler’s Method is the Taylor Method of order 1.

**Example** Use Taylor Method of order 2, 3 and 4 to approximate the solution of the initial-value problem.
\[
y'(t) = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5.
\]

\[
T_2(t_i, y_i) = f(t_i, y_i) + \frac{f'(t_i, y_i)}{2!}h
\]
\[
T_4(t, y(t)) = f(t_i, y_i) + \frac{f'(t_i, y_i)}{2!}h + \frac{f''(t_i, y_i)}{3!}h^2 + \frac{f'''(t_i, y_i)}{4!}h^3
\]
\[ f(t, y(t)) = y - t^2 + 1, \]
\[ f'(t, y(t)) = y' - 2t = y - t^2 + 1 - 2t = y - t^2 - 2t + 1 \]
\[ f''(t, y(t)) = y' - 2t - 2 = y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1 \]
\[ f'''(t, y(t)) = y' - 2t - 2 = y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1 \]

Taylor Method of order 2:

\[ y_0 = 0.5 \]
\[ y_{i+1} = y_i + h \left( y_i - t_i^2 + 1 + \frac{h}{2} (y_i - t_i^2 - 2t_i + 1) \right) \]

Taylor Method of order 3:

\[ y_0 = 0.5 \]
\[ y_{i+1} = y_i + h \left( y_i - t_i^2 + 1 + \frac{h}{2} (y_i - t_i^2 - 2t_i + 1) + \frac{h^2}{6} (y_i - t_i^2 - 2t_i - 1) \right) \]

Taylor Method of order 4:

\[ y_0 = 0.5 \]
\[ y_{i+1} = y_i + h \left( y_i - t_i^2 + 1 + \frac{h}{2} (y_i - t_i^2 - 2t_i + 1) + \frac{h^2}{6} (y_i - t_i^2 - 2t_i - 1) + \frac{h^3}{24} (y_i - t_i^2 - 2t_i - 1) \right) \]

Taylor Method of order 2: \( h = 0.4, h = 0.2, h = 0.1, \) and \( h = 0.05. \)

Let \( h = 0.4, \) comparing Taylor Method of order of 2, 3, and 4.
2. Local Truncation Error:

**Definition** (local truncation error) Consider the difference method:

\[
y_0 = a \\
y_{i+1} = y_i + h \phi(t_i, y_i), \quad i = 0, 1, 2, \ldots, N - 1,
\]

The local truncation error of this difference method is defined as:

\[
\tau_{i+1}(h) = \frac{y(t_{i+1}) - \left( y(t_i) + h \phi(t_i, y(t_i)) \right)}{h} \\
= \frac{y(t_{i+1}) - y(t_i)}{h} - \phi(t_i, y(t_i)), \quad \text{for } i = 0, 1, \ldots, N - 1.
\]

Note that \( \tau_{i+1}(h) \) is a function of \( h \) for \( i = 0, 1, \ldots, N - 1 \).

Local truncation error: using \( \phi(t_i, y(t_i)) = T_n(t_i, y_i) \)

\[
\tau_{i+1}(h) = \frac{y(t_{i+1}) - y(t_i)}{h} - \phi(t_i, y(t_i)), \quad \text{for } i = 0, 1, \ldots, N - 1.
\]

\[
= \frac{1}{h} \left( y(t_i) + y'(t_i)h + \frac{y''(t_i)}{2!}h^2 + \ldots + \frac{y^{(n)}(t_i)}{n!}h^n + \frac{y^{(n+1)}(t_i^*)}{(n+1)!}h^{n+1} - y(t_i) \right) - T_n(t_i, y(t_i))
\]

\[
= \frac{y^{(n+1)}(t_i^*)}{(n+1)!}h^n, \quad \text{where } t_i < t_i^* < t_{i+1}.
\]

If \( |y^{(n+1)}(t)| \leq M \) for \( t \) in \( [a, b] \), then

\[
|\tau_{i+1}(h)| \leq \frac{Mh^n}{(n+1)!} \quad \text{for all } i = 0, 1, \ldots, N.
\]

**Example** Find the local truncation error of Euler’s Method.

Euler’s Method: \( n = 1 \).
\[ |\tau_{i+1}(h)| \leq \frac{Mh^2}{2} \text{ for all } i = 0, 1, \ldots, N. \]

If \(|y''(t)| \leq M\) for all \(t\) in \([a, b]\), then
\[ |\tau_{i+1}(h)| \leq \frac{1}{2}Mh = O(h). \]

**Example** Consider the initial-value problem.
\[ y'(t) = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5. \]

For Euler Method:

Find an upper bound (as small as possible) of the true error \(|y(t_{i+1}) - y_{i+1}|\) for \(i = 0, 1, \ldots\).

Find an upper bound (as small as possible) of the truncation error \(|\tau_{i+1}(h)|\) for \(i = 0, 1, \ldots\).

Compare above error bounds when \(h = 0.1\).

Find \(h\) such that \(|\tau_{i+1}(h)| \leq 0.0001\) for each method.

For Taylor Method of Order 2, 3 and 4:

Find an upper bound (as small as possible) of the truncation error \(|\tau_{i+1}(h)|\) for \(i = 0, 1, \ldots\).

Compare above error bounds when \(h = 0.1\).

Find \(h\) such that \(|\tau_{i+1}(h)| \leq 0.0001\) for each method.

Solution: \(y(t) = (t + 1)^2 - 0.5e^t\)
\[ y'(t) = 2(t + 1) - 0.5e^t, \quad y''(t) = 2 - 0.5e^t, \quad y'''(t) = -0.5e^t \]

Suppose Taylor method of order 2 is used to approximate the solution of this initial-value problem. Find \(h\) so that the magnitudes of all local truncation errors are less than 0.0001.

Euler Method:
\[ |\tau_{i+1}(h)| \leq \frac{0.5e^2h^2}{3!} \leq 0.0001 \text{ for all } i = 0, 1, \ldots, N \]
\[ h^2 \leq \frac{0.0001(6)}{0.5e^2}, \quad h \leq \frac{0.0001(6)}{0.5e^2} = 0.01274372, \text{ let } h = 0.01. \]
\[ |y''(t)| = |2 - 0.5e^t| \leq 2 + \frac{1}{2}e^2 = M, \quad L = 1 \]
\[ |y_{i+1} - y(t_{i+1})| \leq \frac{Mh}{2L} (e^{L(t_{i+1} - t)} - 1) = \frac{2 + \frac{1}{2}e^2}{2(1)} (e^{(1)(2-0)} - 1) = \left(1 + \frac{1}{4}e^2\right)(e^2 - 1)h \]
\[ |\tau_{i+1}(h)| \leq \frac{Mh}{2} = \frac{2 + \frac{1}{2}e^2}{2(1)} h = \left(1 + \frac{1}{4}e^2\right)h \]

When \(h = 0.1\),
\[ |y_{i+1} - y(t_{i+1})| \leq \left( 1 + \frac{1}{4}e^2 \right)(e^2 - 1)(0.1) = 1.82 \]

\[ |\tau_{i+1}(0.1)| \leq \left( 1 + \frac{1}{4}e^2 \right)(0.1) = 0.2847264025 \]

If we want \(|y_{i+1} - y(t_{i+1})| \leq 0.01\), what \(h\) should we choose?

\[ |y_{i+1} - y(t_{i+1})| \leq \frac{Mh}{2L} \left( e^{L(t_{i+1}-a)} - 1 \right) = \frac{2 + \frac{1}{2}e^2}{2(1)}h = (1 + \frac{1}{4}e^2)(e^2 - 1)h \leq 0.01 \]

\[ h \leq \frac{0.01}{(1 + \frac{1}{4}e^2)(e^2 - 1)} = 0.0005497, \text{ let } h = 0.0005 \]

If we want \(|\tau_{i+1}(h)| \leq 0.01\), what \(h\) should we choose?

\[ |\tau_{i+1}(h)| \leq \frac{Mh}{2} = \frac{2 + \frac{1}{2}e^2}{2(1)}h = (1 + \frac{1}{4}e^2)h \leq 0.01 \]

\[ h \leq \frac{0.01}{(1 + \frac{1}{4}e^2)} = 0.003512143557, \text{ let } h = 0.003. \]

Taylor Method of order 2:

\[ |y'''(t)| = 0.5e^t \leq 0.5e^2 = M, \quad |\tau_{i+1}(h)| \leq \frac{0.5e^2h^2}{3!} \]

When \(h = 0.1\),

\[ |\tau_{i+1}(0.1)| \leq \frac{0.5e^2(0.1)^2}{3!} = 0.006157546749 \]

If we want \(|\tau_{i+1}(h)| \leq 0.0001\), what \(h\) should we choose?

\[ |\tau_{i+1}(h)| \leq \frac{0.5e^2}{3!}h^2 \leq 0.0001 \]

\[ h^2 \leq \frac{0.0001}{0.5e^2/3!} \quad \text{or} \quad h \leq \sqrt{\frac{0.0006}{0.5e^2}} = 0.01274371766, \text{ let } h = 0.01 \]