Consider the boundary value problems (BVPs) for the second order differential equation of the form
\[ y'' = f(x,y,y'), \quad a \leq x \leq b, \quad y(a) = \alpha \text{ and } y(b) = \beta. \]
Under what conditions a boundary value problem has a solution or has a unique solution.

1. **Existence and Uniqueness:***
   Suppose that \( f \) is continuous on the set
   \[ D = \{(x,y,y'); a \leq x \leq b, \quad -\infty < y < \infty, \quad -\infty < y' < \infty\} \]
   and the partial derivatives \( f_y \) and \( f_{yy} \) are also continuous on \( D \). If
   (i) \( f_y(x,y,y') > 0 \), for all \( (x,y,y') \) in \( D \), and
   (ii) there exists a constant \( M \) such that
   \[ |f_y(x,y,y')| \leq M \text{ for all } (x,y,y') \text{ in } D, \]
   then the boundary value problem (*) has a unique solution.

**Example** Consider the following boundary value problem:
\[ y'' + e^{-x}y + \sin(y') = 0, \quad 1 \leq x \leq 2, \quad y(1) = y(2) = 0 \]
Determine if the boundary value problem has a unique solution.

Rewrite \( y'' = -e^{-x}y - \sin(y') \) so \( f(x,y,y') = -e^{-x} - \sin(y') \)
Check conditions:
\[ f(x,y,y') = -e^{-x} - \sin(y'), \quad f_y(x,y,y') = xe^{-x}, \quad \text{and} \quad f_{yy}(x,y,y') = -\cos(y') \]
are continuous on
\[ D = \{(x,y,y'); 1 \leq x \leq 2, \quad -\infty < y < \infty, \quad -\infty < y' < \infty\}. \]
(i) \( f_y(x,y,y') = xe^{-x} > 0 \) on \( D \).
(ii) \( |f_y(t,y,y')| = |-\cos(y')| \leq 1 = M. \)
So, the boundary value problem has a unique solution in \( D \).

**Example** Consider the linear boundary value problem:
\[ y'' = p(x)y' + q(x)y + r(x), \quad a \leq x \leq b, \quad y(a) = \alpha, \quad y(b) = \beta \]
Under what condition(s) a linear BVP has a unique solution?
\[ f(x,y,y') = p(x)y' + q(x)y + r(x), \quad f_y(x,y,y') = q(x), \quad f_{yy}(x,y,y') = p(x) \]
are continuous on \( D \) if \( p(x), \quad q(x) \) and \( r(x) \) are continuous for \( a \leq x \leq b. \)
(a) \( f_y(x,y,y') = q(x) > 0 \) for \( a \leq x \leq b. \)
(b) Since \( f_{yy} \) is continuous on \([a, \ b]\), \( f_y \) is bounded.
So, if \( p(x), \ q(x) \) and \( r(x) \) are continuous for \( a \leq x \leq b, \) and \( q(x) > 0 \) for \( a \leq x \leq b, \) then the boundary value problem has a unique solution.

2. **The Linear Shooting Method:**
   Consider the linear boundary value problems of the form:
   \[ y'' = p(x)y' + q(x)y + r(x), \quad a \leq x \leq b, \quad y(a) = \alpha, \quad y(b) = \beta \]
where \( p(x), \ q(x) \) and \( r(x) \) are continuous and \( q(x) > 0 \) for \( a \leq x \leq b. \) Consider the solutions of the
following two initial-value problems:

\[ (** \) \]

(i) \[ y'' = p(x)y' + q(x)y + r(x), \quad a \leq x \leq b, \quad y(a) = \alpha, \quad y'(a) = 0 \]

(ii) \[ y'' = p(x)y' + q(x)y, \quad a \leq x \leq b, \quad y(a) = 0, \quad y'(a) = 1 \]

say, \( y_1(x) \) and \( y_2(x) \). Let \( y(x) \) be the following linear combination of \( y_1(x) \) and \( y_2(x) \):

\[ y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2(x) \]

Then \( y(x) \) is the solution of the boundary value problem. Check:

\[ y''(x) = y_1''(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2''(x) = p(x)y_1'(x) + q(x)y_1(x) + r(x) + \frac{\beta - y_1(b)}{y_2(b)} (p(x)y_2'(x) + q(x)y_2(x)) \]

\[ = p(x) \left( y_1'(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2'(x) \right) + q(x) \left( y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2(x) \right) + r(x) \]

So, \( y(x) \) is a solution of \( y'' = p(x)y' + q(x)y + r(x) \). Check the boundary conditions:

\[ y(a) = y_1(a) + \frac{\beta - y_1(b)}{y_2(b)} y_2(a) = \alpha + \frac{\beta - y_1(b)}{y_2(b)}(0) = \alpha \]

\[ y(b) = y_1(b) + \frac{\beta - y_1(b)}{y_2(b)} y_2(b) = \beta. \]

This suggests that a boundary value problem can be solved by solving two (independent) initial-value problems in (**) .

**Review:** Solve a second-order initial-value problem:

\[ y'' = f(x,y,y'), \quad y(a) = \alpha_0, \quad y'(a) = \alpha_1. \]

Let \( u_1 = y_1 \), and \( u_2 = y_1' \). Then above second-order differential equation for \( y \) becomes the following system of two first-order differential equation in \( u_1 \) and \( u_2 \):

\[
\begin{align*}
&u_1' = u_2 \\
&u_2' = f_1(x,u_1,u_2) \quad , \quad a \leq x \leq b, \quad u_1(a) = \alpha_0, \quad u_2(a) = \alpha_1.
\end{align*}
\]

**Example**  Rewrite the differential equation \( y'' - 2y' + 4y = te^{2t} \) as a system of \( 2 \) 1st-order differential equations.

Set

| \( u_1 = y \) | \( \Rightarrow \) | \( u_1' = y' \) |
| \( u_2 = y' = u_1' \) | \( \Rightarrow \) | \( u_2' = y'' = 2y' - 4y + te^{2t} = -4u_1 + 2u_2 + te^{2t} \) |

The system is:

\[
\begin{align*}
&u_1' = u_2 \\
u_2' = -4u_1 + 2u_2 + te^{2t} \quad \Rightarrow \quad \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ te^{2t} \end{bmatrix}
\end{align*}
\]

**Example**  Rewrite the initial-value problem for the system of \( 2 \) second-order differential equations
Solve the boundary value problem:

\[
\begin{align*}
y''_1 + y'_1 - 2y'_2 + 2y_1 - y_2 &= t \\
y''_2 - 3y'_1 + y_1 + 4y_2 &= 2\sin(t)
\end{align*}
\]

, \( y_1(0) = 1, y_2(0) = -1, y'_1(0) = 2, y'_2(0) = 3 \)

as an initial-value problem for a system of 4 first-order differential equations.

Set

\[
\begin{align*}
&u_1 = y_1 \\
u_2 = y'_1 = u_1' \implies u_1' = u_2 \\
u_3 = y_2 \implies u_2' = y'_1 = -2y_1 + y_2 - y'_1 + 2y'_2 + t = -2u_1 + u_3 - u_2 + 2u_4 + t \\
u_4 = y'_2 = u_3' \implies u_3' = u_4 \\
u_4' = y''_2 = -y_1 - 4y_2 + 3y'_1 + 2\sin(t) = -u_1 - 4u_3 + 3u_2 + 2\sin(t)
\end{align*}
\]

The system of 4 1st-order linear differential equations is:

\[
\begin{align*}
u_1' &= u_2 \\
u_2' &= -2u_1 + u_3 - u_2 + 2u_4 + t, \quad u_1(0) = 1, u_2(0) = 2, u_3(0) = -1, u_4(0) = 3; \\
u_3' &= u_4 \\
u_4' &= -u_1 - 4u_3 + 3u_2 + 2\sin(t)
\end{align*}
\]

or in matrix-vector notation:

\[
\begin{bmatrix}
u_1' \\
u_2' \\
u_3' \\
u_4'
\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0 \\
-2 & -1 & 1 & 2 \\
0 & 0 & 0 & 1 \\
-1 & 3 & -4 & 0
\end{bmatrix} \begin{bmatrix}u_1 \\
u_2 \\
u_3 \\
u_4
\end{bmatrix} + \begin{bmatrix}0 \\
t \\
0 \\
2\sin(t)
\end{bmatrix}, \quad \begin{bmatrix}u_1(0) \\
u_2(0) \\
u_3(0) \\
u_4(0)
\end{bmatrix} = \begin{bmatrix}1 \\
2 \\
-1 \\
3
\end{bmatrix}
\]

**Example** Solve the boundary value problem:

\[
y'' = -\frac{4}{x}y' - \frac{2}{x^2}y + \frac{2}{x^2}\ln x, \quad 1 \leq x \leq 2, \quad y(1) = \frac{1}{2}, \quad y(2) = \ln 2
\]

Exact solution is:

\[
y(x) = -\frac{1}{x^2}\left(2 - 4x + \frac{3}{2}x^2 - x^2\ln x\right).
\]

Note that it is a linear boundary value problem where \( p(x) = -\frac{4}{x}, \quad q(x) = -\frac{2}{x^2}, \quad r(x) = \frac{2}{x^2}\ln x \)

continuous on \([1,2]\).

Since \( q(x) \not\geq 0 \), we cannot say if this boundary value problem has a unique solution. Now we solve the following two initial-value problems:

(i) \[
y''_1 = -\frac{4}{x}y'_1 - \frac{2}{x^2}y_1 + \frac{2}{x^2}\ln x, \quad 1 \leq x \leq 2, \quad y_1(1) = \frac{1}{2}, \quad y'_1(1) = 0
\]

(ii) \[
y''_2 = -\frac{4}{x}y'_2 - \frac{2}{x^2}y_2, \quad 1 \leq x \leq 2, \quad y_2(1) = 0, \quad y'_2(1) = 1
\]

Set up the initial-value problem for a system of 4 1st-order differential equations:
\[
\begin{aligned}
  u_1' &= u_2 \\
  u_2' &= -\frac{4}{x} u_2 - \frac{2}{x^2} u_1 + \frac{2}{x^2} \ln x \\
  u_3' &= u_4 \\
  u_4' &= -\frac{4}{x} u_4 - \frac{2}{x^2} u_3
\end{aligned}
\]

\[u_1(1) = \frac{1}{2}, \quad u_2(1) = 0, \quad u_3(1) = 0, \quad u_4(1) = 1.\]
Shooting Method: $y'' = -4/x^2 y - 2/x^2 y + 2*ln(x)/x^2$, $y(1) = 1/2, y(2) = \ln 2$
MatLab program for this example:

clf
alpha=1/2;
beta=log(2);
a=1;
b=2;
[xv,yv]=ode45('funsysa',[a b],[alpha;0;0;1]);
plot(xv,yv(:,1),'r-.',xv,yv(:,3),'m–')
hold
n=length(yv(:,1));
y1n=yv(n,1);
y2n=yv(n,3);
yvsol=yv(:,1)+(beta-y1n)/y2n*yv(:,3);
trueol=-1./(xv.^2).*(2-4*xv+3/2*xv.^2-xv.^2.*log(xv));
plot(xv,yvsol,'b-',xv,trueol,'k-')
title('Shooting Method: y”=-4/x*y-2/x^2*y+2*ln(x)/x^2, y(1)=1/2,y(2)=ln2')

MatLab function: funsysa.m

function yv=funsysa(t,y);
yv(1,1)=y(2,1);
yv(2,1)=-4/t*y(2,1)-2/(t^2)*y(1,1)+2*log(t)/(t^2);
yv(3,1)=y(4,1);
yv(4,1)=-4/t*y(4,1)-2/(t^2)*y(3,1);

Notes: ode45.m is a MatLab building-in function which solves initial-value problems for systems of n first-order differential equations:

\[
\begin{align*}
  u'_1 &= f_1(x, u_1, \ldots, u_n) \\
  & \vdots \\
  u'_n &= f_n(x, u_1, \ldots, u_n)
\end{align*}
\]

It is called by:
[xv,uv]=ode45('funsysa',[a b],[alpha1, \ldots, alpha_n]); where funsysa.m is a user-provided Matlab program that
evaluates functions of the system at $x$. The outputs are the vector $x v = [x_0, x_1, \ldots, x_{N-1}, x_N]$ and

$$u v = \begin{bmatrix} u_{10} & \cdots & u_{n0} \\ u_{11} & \cdots & u_{n1} \\ \vdots & \ddots & \vdots \\ u_{1N} & \cdots & u_{nN} \end{bmatrix}.$$ That is, for $a \leq x \leq b$,

$$y_1(x) \approx \{u_{10}, u_{11}, \ldots, u_{1N}\}, \ldots, y_n(x) \approx \{u_{n0}, u_{n1}, \ldots, u_{nN}\}.$$