5.3 - Higher-Order Taylor Method

Consider solving the initial-value problem for ordinary differential equation:

\[ (*) \quad y'(t) = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha. \]

Let \( y(t) \) be the unique solution of the initial-value problem. In Section 5.2, Euler’s Method, a numerical method, is introduced to compute a set \( \{ y_k \}_{k=0}^N \) where \( y_k \approx y(t_k) \) and \( a = t_0 < t_1 < \ldots < t_{N-1} < t_N = b. \) Recall that Euler’s Method is derived using a Taylor polynomial of degree 1. Will a numerical approximation be better if a Taylor polynomial of degree 2 or degree 3 is used? In this section, we derive a numerical method using Taylor Method of order \( n \) where \( n = 1, 2, \ldots. \)

Assume that the initial-value problem is well-posed. Let \( h = \frac{b-a}{N} \), where \( N \) is a positive integer. Define \( t_k = t_{k-1} + h \) or \( t_k = a + k h. \)

1. **Taylor Method of order \( n \) (\( n \)-th Order of Taylor Method):**

We first express the solution \( y(t_{i+1}) \) to the initial-value problem (*) by its \( n \)th Taylor polynomial and corresponding remainder centered at \( t_i: \)

\[
\begin{align*}
    y(t_{i+1}) &= y(t_i) + y'(t_i) h + \frac{y''(t_i)}{2!} h^2 + \ldots + \frac{y^{(n)}(t_i)}{n!} h^n + \frac{y^{(n+1)}(t_i^*)}{(n+1)!} h^{n+1}, \\
    \text{where } t_i &\leq t_i^* \leq t_{i+1} \\
    &= y(t_i) + f(t_i, y(t_i)) h + \frac{f'(t_i, y(t_i))}{2!} h^2 + \ldots + \frac{f^{(n-1)}(t_i, y(t_i))}{n!} h^n + \frac{f^{(n)}(t_i, y(t_i))}{(n+1)!} h^{n+1} \\
    &= y(t_i) + h \left( \frac{f(t_i, y(t_i))}{2!} h + \ldots + \frac{f^{(n-1)}(t_i, y(t_i))}{n!} h^n \right) + \frac{f^{(n)}(t_i, y(t_i))}{(n+1)!} h^{n+1}.
\end{align*}
\]

Define

\[ T_n(t, y(t)) = f(t, y(t)) + \frac{f'(t, y(t))}{2!} h + \ldots + \frac{f^{(n-1)}(t, y(t))}{n!} h^{n-1}. \]

**Taylor Method of order \( n \) (\( n \)-th order of Taylor Method):**

\[ y_0 = \alpha \]

\[ y_i+1 = y_i + h T_n(t_i, y_i), \quad i = 0, 1, 2, \ldots, N-1, \]

Notes:

a. \( T_1(t, y(t)) = f(t, y(t)) \). So, Euler’s Method is the Taylor Method of order 1.

b. \( T_n(t, y(t)) = T_{n-1}(t, y(t)) + \frac{f^{(n-1)}(t, y(t))}{n!} h^{n-1}. \) If \( h \) is small, for large \( n \), \( T_n(t, y(t)) \approx T_{n-1}(t, y(t)) \), that means \( T_n(t, y(t)) \) will not improve much the approximation.

**Example** Use Taylor Method of order 2, 3 and 4 to approximate the solution of the initial-value problem.

\[ y'(t) = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5. \]
\[ T_2(t_i, y_i) = f(t_i, y_i) + \frac{f'(t_i, y_i)}{2!} h \]
\[ T_3(t_i, y_i) = T_2(t_i, y_i) + \frac{f''(t_i, y_i)}{3!} h^2 \]
\[ T_4(t, y(t)) = T_3(t_i, y_i) + \frac{f'''(t_i, y_i)}{4!} h^3 \]

\[
f(t, y(t)) = y - t^2 + 1, \\
f'(t, y(t)) = y' - 2t = y - t^2 + 1 - 2t = y - t^2 - 2t + 1 \\
f''(t, y(t)) = y'' - 2t - 2 = y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1 \\
f'''(t, y(t)) = y''' - 2t - 2 = y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1
\]

Taylor Method of order 2:
\[
y_0 = 0.5 \\
y_{i+1} = y_i + h \left( y_i - t_i^2 + 1 + \frac{h}{2} (y_i - t_i^2 - 2t_i + 1) \right)
\]

Taylor Method of order 3:
\[
y_0 = 0.5 \\
y_{i+1} = y_i + h \left( y_i - t_i^2 + 1 + \frac{h}{2} (y_i - t_i^2 - 2t_i + 1) + \frac{h^2}{6} (y_i - t_i^2 - 2t_i - 1) \right)
\]

Taylor Method of order 4:
\[
y_0 = 0.5 \\
y_{i+1} = y_i + h \left( y_i - t_i^2 + 1 + \frac{h}{2} (y_i - t_i^2 - 2t_i + 1) + \frac{h^2}{6} (y_i - t_i^2 - 2t_i - 1) + \frac{h^3}{24} (y_i - t_i^2 - 2t_i - 1) \right)
\]

Taylor Method of order 2: \( h = 0.4, \ h = 0.2, \ h = 0.1, \) and \( h = 0.05. \)

In Matlab:
```
>> fun=@(t,y) y-t.^2+1;  
>> fun1=@(t,y) y-t.^2-2*t+1;  
>> [tv yv,n]=tayfun2(fun,fun1,0,2,0.5,0.4);  
>> plot(tv,yv,'k-o')  
>> hold on  
>> [tv yv,n]=tayfun2(fun,fun1,0,2,0.5,0.2);  
>> plot(tv,yv,'r-*')  
>> [tv yv,n]=tayfun2(fun,fun1,0,2,0.5,0.1);  
>> plot(tv,yv,'b-x')  
>> [tv yv,n]=tayfun2(fun,fun1,0,2,0.5,0.05);  
>> plot(tv,yv,'g-+')  
>> title('dy/dt = y-t^2+1, y(0)=1/2, t in [0,2]')  
>> hold off
```
Let $h = 0.4$, comparing Taylor Method of order of 2, 3, and 4.

In MatLab:

```matlab
>> fun=@(t,y) y-t.^2+1;
>> fun1=@(t,y) y-t.^2-2*t+1;
>> fun2=@(t,y) y-t.^2-2*t-1;
(Note that fun3=fun2)
>> [tv yv,n]=tayfun2(fun,fun1,0,2,0.5,0.4);
>> plot(tv,yv,’k-o’)
>> hold on
>> [tv yv,n]=tayfun3(fun,fun1,fun2,0,2,0.5,0.4);
>> plot(tv,yv,’r-*’)
>> [tv yv,n]=tayfun4(fun,fun1,fun2,fun2,0,2,0.5,0.4);
>> plot(tv,yv,’b-x’)
>> title(‘$dy/dt=y-t^2+1$, $y(0)=1/2$, $t$ in [0,2]’)
>> hold off
```
2. Local Truncation Error:

**Definition (local truncation error)** Consider the difference method:

\[ y_0 = a \]
\[ y_{i+1} = y_i + h \phi(t_i, y_i), \quad i = 0, 1, 2, \ldots, N - 1, \]

The local truncation error of this difference method is defined as:

\[
\tau_{i+1}(h) = \frac{y(t_{i+1}) - (y(t_i) + h \phi(t_i, y(t_i)))}{h} = \frac{y(t_{i+1}) - y(t_i)}{h} - \phi(t_i, y(t_i)), \quad \text{for } i = 0, 1, \ldots, N - 1.
\]

Note that \( \tau_{i+1}(h) \) is a function of \( h \) for \( i = 0, 1, \ldots, N - 1 \).

When \( \phi(t_i, y(t_i)) = T_n(t_i, y_i) \), the local truncation error can be derived as follows.

\[
\tau_{i+1}(h) = \frac{y(t_{i+1}) - y(t_i)}{h} - \phi(t_i, y(t_i)), \quad \text{for } i = 0, 1, \ldots, N - 1.
\]
\[
= \frac{1}{h} \left( y(t_i) + y'(t_i)h + \frac{y''(t_i)}{2!}h^2 + \ldots + \frac{y^{(n)}(t_i)}{n!}h^n + \frac{y^{(n+1)}(t_i^*)}{(n + 1)!}h^{n+1} - y(t_i) \right) - T_n(t_i, y(t_i))
\]
\[
= \frac{y^{(n+1)}(t_i^*)}{(n + 1)!}h^n, \quad \text{where } t_i < t_i^* < t_{i+1}.
\]

If \( |y^{(n+1)}(t)| \leq M \) for \( t \) in \([a, b]\), then

\[
|\tau_{i+1}(h)| \leq \frac{Mh^n}{(n + 1)!} \quad \text{for all } i = 0, 1, \ldots, N.
\]

**Example** Find the local truncation error of Euler’s Method.

Euler’s Method: \( n = 1 \).
If $|y''(t)| \leq M$ for all $t$ in $[a, b]$, then

$$|\tau_{i+1}(h)| \leq \frac{1}{2} Mh = O(h).$$

**Example**  
Consider the initial-value problem.

$$y'(t) = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5.$$

Suppose we know that $y(t) = (t + 1)^2 - 0.5e^t$.

(1) **For Euler Method:**

(i) Find an upper bound (as small as possible) of the true error $|y(t_{i+1}) - y_{i+1}|$ for $i = 0, 1, \ldots$ in term of $h$.

(ii) Find an upper bound (as small as possible) of the truncation error $|\tau_{i+1}(h)|$ for $i = 0, 1, \ldots$ in term of $h$.

(iii) Compare above error bounds when $h = 0.1$.

(iv) Find $h$ such that $|y(t_{i+1}) - y_{i+1}| \leq 0.01$.

(v) Find $h$ such that $|\tau_{i+1}(h)| \leq 0.01$.

(2) **For Taylor Method of Order 2, 3 and 4:**

(i) Find an upper bound (as small as possible) of the truncation error $|\tau_{i+1}(h)|$ for $i = 0, 1, \ldots$ in terms of $h$.

(ii) Compare above error bounds when $h = 0.1$.

(iii) Find $h$ such that $|\tau_{i+1}(h)| \leq 0.0001$ for each method.

$$y(t) = (t + 1)^2 - 0.5e^t, \quad y'(t) = 2(t + 1) - 0.5e^t, \quad y''(t) = 2 - 0.5e^t, \quad y'''(t) = -0.5e^t$$

$$|y''(t)| = |2 - 0.5e^t| \leq 2 + \frac{1}{2}e^2 = 5.69452805 = M, \quad L = 1$$

Or from the graph of $|y''(t)| = |2 - 0.5e^t|$, we have $M = |y''(2)| = |2 - 0.5e^2| = 1.69452805$. Let $M = 1.7$.

(1) **For Euler Method:**
(ii) \[ |\tau_{i+1}(h)| \leq \frac{Mh}{2} = \frac{(1.7)h}{2} = 0.85h \]

(iii) For \( h = 0.1 \),
\[ |y_{i+1} - y(t_{i+1})| \leq 0.85(e^2 - 1)(0.1) = 0.543069768 \text{ and} \]
\[ |\tau_{i+1}(h)| \leq 0.85(0.1) = 0.085 \]
\[ |\tau_{i+1}(h)| \text{ is smaller than } |y_{i+1} - y(t_{i+1})|. \]

(iv) \[ |y(t_{i+1}) - y_{i+1}| \leq 0.85(e^2 - 1)h \leq 0.01, \ h \leq \frac{0.01}{0.85(e^2 - 1)} = 1.84138403 \times 10^{-3}, \text{ let } h = 0.0018. \]

(v) \[ |\tau_{i+1}(h)| \leq 0.85h \leq 0.01, \ h \leq \frac{0.01}{0.85} = 1.17647059 \times 10^{-2}, \text{ let } h = 0.01176. \]

(2) For Taylor Method of order 2, 3, and 4,

(i) \( n = 2 \), \[ |\tau_{i+1}(h)| \leq \frac{Mh^2}{3!} = \frac{(1.7)h^2}{6} \]

(ii) Let \( h = 0.1 \).
\[ n = 2, \ |\tau_{i+1}(h)| \leq \frac{(1.7)(0.1)^2}{6} = 2.8333333 \times 10^{-3} \]
\[ n = 3, \ |\tau_{i+1}(h)| \leq \frac{(1.7)(0.1)^3}{4!} = 7.0833333 \times 10^{-5} \]
\[ n = 4, \ |\tau_{i+1}(h)| \leq \frac{(1.7)(0.1)^4}{5!} = 1.41666667 \times 10^{-6} \]
\[ |\tau_{i+1}(h)| \text{ is getting smaller as } n \text{ is getting larger.} \]

(iii) \[ n = 2, \ |\tau_{i+1}(h)| \leq \frac{(1.7)h^2}{6} \leq 0.0001, \ h \leq \sqrt{\frac{0.0001(6)}{1.7}} = 1.87867287 \times 10^{-2}, \text{ let } h = 0.018 \]
\[ n = 3, \ |\tau_{i+1}(h)| \leq \frac{(1.7)h^3}{24} \leq 0.0001, \ h \leq \sqrt{\frac{0.0001(24)}{1.7}} = 0.112181379, \text{ let } h = 0.1. \]
\[ n = 4, \ |\tau_{i+1}(h)| \leq \frac{(1.7)h^4}{120} \leq 0.0001, \ h \leq \sqrt{\frac{0.0001(120)}{1.7}} = 0.289856525, \text{ let } h = 0.28. \]

Exercises:

1. For each of the initial-value problems,
   (1) identify the function \( f(t,y) \);
   (2) compute \( \frac{df(t,y(t))}{dt} \);
   (3) apply the second order Taylor Method with \( N = 2 \) without using MatLab program tayfun2.m; and
   (4) apply the second order Taylor Method with \( h = 0.2 \) using MatLab program tayfun2.m.
      a. \( y' = \frac{e^t}{y}, \ 0 \leq t \leq 1, y(0) = 1 \)
      b. \( y' + ty = ty^2, \ 0 \leq t \leq 2, y(0) = 0.5 \)
c. \( y' = te^{ty} - 1, \ 0 \leq t \leq 2, \ y(0) = -1 \)
d. \( y' = e^{2t} + (1 + \frac{5}{2} e^t)y + y^2, \ 0 \leq t \leq 1, \ y(0) = -1 \)

2. For each of the initial-value problems,
   (1) identify the function \( f(t,y) \);
   (2) compute \( \frac{df(t,y(t))}{dt}, \ \frac{d^2f(t,y(t))}{dt^2}, \ \text{and} \ \frac{d^3f(t,y(t))}{dt^3} \);
   (3) apply the Taylor Method of order 4 with \( N = 2 \) without using MatLab program tayfun4.m; and
   (4) apply the Taylor Method of order 4 with \( h = 0.1 \) using MatLab program tayfun4.m.
   a. \( y' + 2y^2 = t^2 - 1, \ 0 \leq t \leq 1, \ y(0) = 0 \)
   b. \( y' = \sin(t) - y, \ \pi \leq t \leq 2\pi, \ y(\pi) = 1 \)
   c. \( y' + \frac{4y}{t} = t^4, \ 1 \leq t \leq 2, \ y(1) = 1 \)
   d. \( y' = y - t, \ 0 \leq t \leq 1, \ y(0) = 2 \)

3. Let \( y(t) \) be the solution of the initial-value problem: \( y' = f(t,y), \ a \leq t \leq b, \ y(a) = a \). The graph of \( y(t) \) is given below. For each problem, estimate graphically \( y_1 \) and \( y_2 \) obtained by Euler’s Method with \( h = 0.25 \) and \( h = 0.5 \), respectively. Explain graphically how \( y_1 \) and \( y_2 \) are derived.

4. Consider using the 2nd order Taylor Method. For each of the following initial-value problems,
   (1) find an upper bound (as small as possible) of the truncation error \( |r_{i+1}(h)| \) (use the true solution \( y(t) \) to find \( M \)) and \( |r_{i+1}(h)| \) with given \( h \); and
   (2) find \( h \) such that \( |r_{i+1}(h)| \leq 0.01 \).
   a. \( y' = \frac{e^t}{y}, \ 0 \leq t \leq 1, \ y(0) = 1, \ y(t) = \sqrt{2e^t - 1}, \ h = 0.1 \)
   b. \( y' + ty = ty^2, \ 0 \leq t \leq 2, \ y(0) = 0.5, \ y(t) = (1 + e^{t^2/2})^{-1}, \ h = 0.2 \)
   c. \( y' = te^{ty} - 1, \ 0 \leq t \leq 2, \ y(0) = -1, \ y(t) = -t - \ln(e - t^2/2), \ h = 0.2 \)
   d. \( y' = \sin(t) - y, \ \pi \leq t \leq 2\pi, \ y(\pi) = 1, \ y(t) = \frac{1}{2} e^{\pi t} + \frac{1}{2} \sin(t) - \frac{1}{2} \cos(t), \ h = \frac{\pi}{10} \)
e. $y' + \frac{4y}{t} = t^4$, $1 \leq t \leq 2$, $y(1) = 1$, $y(t) = \frac{1}{9} t^5 + \frac{8}{9} t^{-4}$, $h = 0.1$

5. Consider the initial-value problem: $y' = \frac{t}{y}$, $0 \leq t \leq 5$, $y(0) = 1$. The true solution is $y(t) = \sqrt{t^2 + 1}$.

For each of the methods: Euler’s Method, the 2nd order Taylor Method and the 4th order Taylor Method,

(1) find an upper bound (as small as possible) of the truncation error $|\tau_{i+1}(h)|$ (use the true solution $y(t)$ to find $M$) and $|\tau_{i+1}(h)|$ with $h = 0.1$; and

(2) find $h$ such that $|\tau_{i+1}(h)| \leq 0.01$. 