The Minimum Rank of a Sign Pattern Matrix Over a Finite Field

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Abstract

A sign pattern matrix is a matrix whose entries are in the set \{+, −, 0\}. The minimum rank of a sign pattern matrix \(S\) is equal to the smallest rank among the ranks of all the matrices (over a given field) whose entries have the same sign pattern as \(S\). For this project, we will study the minimum rank of 2x2 matrices over the field \(\mathbb{R}\) and the finite field \(\mathbb{Z}_p\) (where \(p\) is a prime), and analyze the effects on the minimum rank and sign pattern as the field is changed to the finite field of \(\mathbb{Z}_p\) as well as restricted to symmetric and then symmetric positive definite and positive semi-definite matrices.

1. Introduction

The history of matrices goes back to ancient times! But the term "matrix" was not applied to the concept until 1850. The term "matrix" for such arrangements was introduced in 1850 by James Joseph Sylvester. "Matrix" is the Latin word for womb, and it retains that sense in English. It can also mean more generally any place in which something is formed or produced.

The origins of mathematical matrices lie with the study of systems of simultaneous linear equations. An important Chinese text from between 300 BC and AD 200, Nine Chapters of the Mathematical Art (Chiu Chang Suan Shu), gives the first known example of the use of matrix methods to solve simultaneous equations.

Combinatorial matrix theory, encompassing connections between linear algebra, graph theory, and combinatorics, has emerged as a vital area of research over the last few decades, having applications to fields as diverse as biology, chemistry, economics, and computer engineering. Specifically, the eigenvalues of a matrix of data play a vital role in many applications. Sometimes the entries of a data matrix are not known exactly. This has led to several areas of qualitative matrix theory, including the study of sign pattern matrices. A sign pattern matrix is a matrix whose entries are in the set \{+, −, 0\}. Early work on sign pattern matrices arose from questions in economics and answered the question of what sign patterns require stability, and on sign nonsingularity and sign solvability.

Since their first appearance in ancient China, matrices and sign pattern matrices have remained important mathematical tools. Today, sign pattern matrices are used not simply for solving systems of simultaneous linear equations, but also for describing the quantum mechanics of atomic structure, designing computer game graphics, analyzing relationships, and even plotting complicated dance steps!

2. Preliminary Information

In order to have a better understanding of the information presented in the paper, we first define some basic properties of matrices and some that may not be well known. Throughout the paper, we consider 2x2 matrices of the form \(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\). The determinant of \(A\) is defined by \(ad − bc\).

Definition 1: Sign Pattern Matrix:

A sign pattern matrix is a matrix whose entries are in \{0, +, −\}.
Example: \[
\begin{bmatrix}
0 & + \\
+ & 0
\end{bmatrix}
\text{ or } \begin{bmatrix}
+ & - \\
- & +
\end{bmatrix}
\]

I will refer to a sign pattern matrix as \(S\).

**Definition 2: Rank of a Matrix:**

The rank of a matrix is defined by the maximum number of rows (or columns) of the matrix which are linearly independent.

There are 3 possible ranks for a \(2 \times 2\) matrix.

1. \(\text{rank}(A) = 0\): There is only one such matrix and it is known as the "zero matrix" \[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\].

2. \(\text{rank}(A) = 1\): A nonzero matrix has rank 1 if and only if its determinant equals 0.

   Example: \[
\begin{bmatrix}
3 & 3 \\
2 & 2
\end{bmatrix}
\] where \(ad - bc = 3 \times 2 - 3 \times 2 = 0\).

3. \(\text{rank}(A) = 2\): A nonzero matrix has rank 2 if and only if its determinant does not equal 0.

   Example: \[
\begin{bmatrix}
2 & 3 \\
3 & 2
\end{bmatrix}
\] where \(ad - bc = 2 \times 2 - 3 \times 3 = -5 \neq 0\).

**Definition 3: Minimum Rank of a \(2 \times 2\) Sign Pattern Matrix:**

The minimum rank (\(mr(S)\)) of a sign pattern matrix \(S\) is defined by the minimum of the ranks of the real matrices whose entries have signs equal to the corresponding entries of \(S\).

Let \(S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\) be a sign pattern matrix.

Example: Let \(S_1 = \begin{bmatrix} + & + \\ 0 & 0 \end{bmatrix}\). \(mr(S) = 1\) because \(ad - bc = a(0) - b(0) = 0\).

Example: Let \(S_2 = \begin{bmatrix} + & + \\ + & + \end{bmatrix}\). It may not be easy \(mr(S_2) = 1\). Note that if there exists one matrix in sign pattern \(S\) whose rank is 1 then \(mr(S) = 1\). Because \(\det\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = 0\) and \(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\) has a sign pattern \(S\), \(mr(S) = 1\).

**Definition 4: Non-Singular Matrix:**

An \(n \times n\) matrix \(A\) is said to be **nonsingular**, or **invertible**, if there exists an \(n \times n\) matrix \(B\) such that \(AB = BA = I_n\) (where \(I_n\) is the \(n \times n\) identity matrix). The matrix \(B\) is called the **inverse** of \(A\). Otherwise, \(A\) is said to be **singular**, or **noninvertible**.
Example: Let \( A = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix} \) and \( B = \begin{bmatrix} -1 & 3/2 \\ 1 & -1 \end{bmatrix} \). Since \( AB = BA = I_2 \), we conclude that \( B \) is the inverse of \( A \).

**Definition 5: Symmetric Matrix:**

A symmetric matrix is a square matrix which equals to its transpose, i.e., \( A^T = A \).

Example: \( A = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix} \) is a symmetric matrix, or more general \( \begin{bmatrix} a & b \\ b & d \end{bmatrix} \) is a symmetric matrix.

**Definition 6: Symmetric Positive Definite Matrix:**

A real symmetric matrix \( A \) is said to be:

1. positive definite if \( x^T A x > 0 \) for all nonzero \( x \) in \( \mathbb{R}^n \); and
2. positive semi-definite if \( x^T A x \geq 0 \) for all nonzero \( x \) in \( \mathbb{R}^n \).

**Theorem** Let \( A \) be a real symmetric matrix. Then all eigenvalues of \( A \) are positive and \( A \) is nonsingular.

**Proof:** Let \( (\lambda_i, x_i) \) be an eigenpair of \( A \). Then \( x_i^T A x_i = x_i^T (\lambda_i x_i) = \lambda_i (x_i^T x_i) \) and \( \lambda_i = \frac{x_i^T A x_i}{x_i^T x_i} \). By definition \( x_i^T A x_i > 0 \). Because \( x_i \neq 0 \), \( x_i^T x_i = \|x_i\|^2 > 0 \). Therefore, \( \lambda_i > 0 \).

Because \( \det(A) = \Pi_{i=1}^n \lambda_i > 0 \neq 0 \), \( A \) is nonsingular.

### 3. Notations Used

In this section I will reveal the notations used in my paper in order to dispel any confusion or inconsistencies in understanding.

1. The symbol \( S \) will be used to denote a 2x2 sign pattern matrix.
2. The symbol \( S^F \) will be use to denote a sign pattern matrix over the field \( F \), where \( F = \langle \mathbb{R}, \mathbb{Z}_p \rangle \).
3. The symbol \( S^F_B \) will be used to denote a sign pattern matrix over the field \( F \) with the restriction \( B \), where \( B \) can be \( \text{sym} \) (for symmetric matrices), \( +\text{def} \) (for positive definite), or \( k = \mathbb{Z}^+ \).

Example: \( S^\mathbb{R}_1 \) is a sign pattern matrix over the field \( \mathbb{R} \) with labeled as the first sign pattern matrix.

Example: \( S^\mathbb{R}_{\text{sym}-k} \) is a sign pattern matrix for all symmetric matrices over the field \( \mathbb{R} \) with labeled as the \( k \)th sign pattern matrix.

Example: \( S^\mathbb{Z}_p_{+\text{def}-2} \) is a sign pattern matrix for all positive definite \( (+\text{def}) \) over \( \mathbb{Z}_p \) with labeled as the second sign pattern matrix.

4. \( |S^\mathbb{R}| \) denotes as the number of sign pattern matrices over \( \mathbb{R} \).
5. \( mr(S^\mathbb{R}_{\text{sym}}) \) denotes as the minimum rank of the sign pattern \( S^\mathbb{R}_{\text{sym}} \).

### 4. Minimum Rank of Sign Pattern Matrices
4-1.1 \( S^\mathbb{R} \) (Sign Pattern Matrices in \( \mathbb{R} \))

Among all possible 2x2 matrices, there are a total of 81 different sign pattern matrices. Let \( S = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} \).

Three choices for each entry \( s_i \in \{+,-,0\} \) taken four at a time gives \( |S^\mathbb{R}| = 3^4 = 81 \). We will divide 81 different sign pattern matrices into 3 groups by their ranks. Note that among all nonzero matrices if there exists one matrix in sign pattern \( S \) whose rank is 1 then \( mr(S) = 1 \) and if all matrices in sign pattern \( S \) whose ranks are 2 then \( mr(S) = 2 \). Note also that for a nonzero matrix \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), rank \( A \) = 1 if and only if \( \det(A) = ad - bc = 0 \) and rank \( A \) = 2 if and only if \( \det(A) = ad - bc \neq 0 \). For \( ad - bc = 0 \), we have the following possible choices of \( a,b,c \) and \( d \):

*** Set a table here for all choice of \( a,b,c,d \) to have \( ad-bc=0 ***

Based on all possible choices of \( a,b,c \) and \( d \) analyzed in above table, the 81 sign pattern matrices can be grouped as follows.

1. \( mr(S^\mathbb{R}) = 0 \):

   There is only 1 sign pattern matrix with minimum rank 0, that is \( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) (4 zeros). For simplicity I will exclude this sign pattern in later discussion.

2. \( mr(S^\mathbb{R}) = 1 \):

   There are 32 sign pattern matrices with \( mr(S^\mathbb{R}) = 1 \) and can be grouped into the following 4 basic sign patterns.

   a. \( S_1^\mathbb{R} = \begin{bmatrix} 0 & 0 \\ \pm & 0 \end{bmatrix} \) where any 3 positions are 0 (3 zeros).

   b. \( S_2^\mathbb{R} = \begin{bmatrix} 0 & \pm \\ \pm & 0 \end{bmatrix} \) or \( \begin{bmatrix} 0 & \pm \\ 0 & 0 \end{bmatrix} \) or \( \begin{bmatrix} \pm & 0 \\ 0 & \pm \end{bmatrix} \) where a column or a row is zero.

   c. \( S_3^\mathbb{R} = \begin{bmatrix} \pm & \mp \\ \mp & \pm \end{bmatrix} \) or \( \begin{bmatrix} \mp & \pm \\ \pm & \mp \end{bmatrix} \) or \( \begin{bmatrix} \mp & \pm \\ \pm & \mp \end{bmatrix} \) where any 2 or 2 + in a row, a column, or diagonal of a nonzero matrix.

   d. \( S_4^\mathbb{R} = \begin{bmatrix} + & + \\ + & + \end{bmatrix} \) or \( \begin{bmatrix} - & + \\ - & + \end{bmatrix} \) where all are + or all are −.

3. \( mr(S^\mathbb{R}) = 2 \):

   There are 48 sign pattern matrices with \( mr(S^\mathbb{R}) = 2 \) and can be reduced to the following 3 sign patterns.

   a. \( S_5^\mathbb{R} = \begin{bmatrix} 0 & \pm \\ \pm & \pm \end{bmatrix} \) where any one entry is 0.

   b. \( S_6^\mathbb{R} = \begin{bmatrix} \pm & 0 \\ 0 & \pm \end{bmatrix} \) or \( \begin{bmatrix} 0 & \pm \\ \pm & 0 \end{bmatrix} \) where one diagonal is zero.
c. \( S_{\text{sym}}^R = \begin{bmatrix} + & + \\ + & - \\ - & + \\ - & - \end{bmatrix} \) or \( \begin{bmatrix} - & - \\ - & + \\ + & - \\ + & + \end{bmatrix} \) where any 3 entries are + or -.

### 4-1.2 \( S_{\text{sym}}^R \) (Symmetric Sign Pattern Matrices in \( \mathbb{R} \))

Consider now all \( 2 \times 2 \) symmetric real matrices \( \begin{bmatrix} a & b \\ b & d \end{bmatrix} \). Because the sign of the off-diagonal entries are always the same, \( |S_{\text{sym}}^R| = 3^3 = 27 \). Using the same analysis as given in 1-1, we can divide these 27 sign pattern matrices into the following 3 groups.

1. \( mr(S_{\text{sym}}^R) = 0 \):

Only the sign pattern matrix with 0 rank is \( S_{\text{sym}}^R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \).

2. \( mr(S_{\text{sym}}^R) = 1 \):

a. \( S_{\text{sym}}^{R-2} = \begin{bmatrix} + & + \\ + & - \\ - & + \\ - & - \end{bmatrix} \) where \( b \) is + and \( |S_{\text{sym}}^{R-1}| = 2 \).

b. \( S_{\text{sym}}^{R-3} = \begin{bmatrix} + & - \\ - & + \\ - & - \\ - & - \end{bmatrix} \) where \( b \) is - and \( |S_{\text{sym}}^{R-2}| = 2 \).

c. \( S_{\text{sym}}^{R-4} = \begin{bmatrix} 0 & 0 \\ 0 & \pm \\ \pm & 0 \\ 0 & 0 \end{bmatrix} \) where \( b = 0 \) and one other 0. \( |S_{\text{sym}}^{R-3}| = 4 \).

3. \( mr(S_{\text{sym}}^R) = 2 \): a total of 18 sign pattern matrices

a. \( S_{\text{sym}}^{R-5} = \begin{bmatrix} \mp & + \\ + & \mp \\ + & 0 \\ + & 0 \end{bmatrix} \) or \( \begin{bmatrix} 0 & + \\ + & + \\ + & \pm \\ \pm & + \end{bmatrix} \) or \( \begin{bmatrix} \pm & + \\ + & 0 \\ + & + \\ 0 & \pm \end{bmatrix} \) where \( b \) is + and \( |S_{\text{sym}}^{R-4}| = 7 \).

Example: 
\( \begin{bmatrix} - & + \\ + & + \\ + & + \\ + & + \end{bmatrix} \).

b. \( S_{\text{sym}}^{R-6} = \begin{bmatrix} \mp & - \\ - & \mp \\ - & \pm \\ - & 0 \end{bmatrix} \) or \( \begin{bmatrix} 0 & - \\ - & - \\ - & - \\ - & - \end{bmatrix} \) or \( \begin{bmatrix} \pm & - \\ - & 0 \\ 0 & - \\ - & 0 \end{bmatrix} \) where \( b \) is - and \( |S_{\text{sym}}^{R-5}| = 7 \).

Example: 
\( \begin{bmatrix} - & - \\ - & + \\ - & + \\ - & + \end{bmatrix} \).

c. \( S_{\text{sym}}^{R-7} = \begin{bmatrix} \pm & 0 \\ 0 & \mp \\ 0 & 0 \\ 0 & \pm \end{bmatrix} \) where \( b \) is 0 and \( |S_{\text{sym}}^{R-6}| = 4 \).

### 4-1.3 \( S_{\text{+def}}^R \) (Positive Definite Sign Pattern Matrices in \( \mathbb{R} \))

Now we consider \( 2 \times 2 \) symmetric and positive definite matrices. Because all positive definite matrices are nonsingular, \( mr(S) = 2 \). In this section, we only need to examine the 18 sign pattern matrices in the third group.
in 1-2. We first further study properties of eigenvalues of a $2 \times 2$ positive definite matrix.

Let $S^R = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ and $\lambda_1, \lambda_2$ be eigenvalues of $S^R$. Because

$$\det(S^R - \lambda I_2) = 0 \iff \begin{cases} \lambda^2 - (a + d)\lambda + ad - b^2 = 0 \\ \text{trace det}(S^R) \\ (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0 \end{cases},$$

$\lambda_1\lambda_2 = ad - b^2$ and $\lambda_1 + \lambda_2 = a + d$. Since both $\lambda_1$ and $\lambda_2$ are positive, we have the following two conditions:

(i) $\lambda_1 + \lambda_2 = a + d > 0$; and

(ii) $\lambda_1\lambda_2 = ad - b^2 > 0$.

We examine the following 18 possible symmetric matrices with above two conditions:

1. $S^R_{\text{sym-5}} = \begin{bmatrix} + & + \\ + & - \end{bmatrix}$ or $\begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$ or $\begin{bmatrix} \pm & + \\ + & 0 \end{bmatrix}$

2. $S^R_{\text{sym-6}} = \begin{bmatrix} + & - \\ - & - \end{bmatrix}$ or $\begin{bmatrix} 0 & - \\ - & 0 \end{bmatrix}$ or $\begin{bmatrix} \pm & - \\ - & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & - \\ - & 0 \end{bmatrix}$

3. $S^R_{\text{sym-7}} = \begin{bmatrix} \pm & 0 \\ 0 & \pm \end{bmatrix}$ or $\begin{bmatrix} \mp & 0 \\ 0 & \pm \end{bmatrix}$

First, sign pattern matrices $\begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & - \\ - & 0 \end{bmatrix}$ are eliminated by condition (i) because $a + d > 0$. Secondly, by screening the condition (ii): $ad - b^2 > 0$, sign pattern matrices $\begin{bmatrix} \mp & + \\ + & 0 \end{bmatrix}$ and $\begin{bmatrix} \mp & - \\ - & 0 \end{bmatrix}$ all fail due to $ad < 0$. Now, we have 9 possibilities left:

$\begin{bmatrix} 0 & + \\ + & \pm \end{bmatrix}$, $\begin{bmatrix} \pm & + \\ + & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & - \\ - & \pm \end{bmatrix}$, $\begin{bmatrix} \pm & - \\ - & 0 \end{bmatrix}$. They again fail due to $-b^2 < 0$. Leaving only 1 possibility:

$S^R_{\text{def}} = \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}$.

This matrix is also a diagonal dominate matrix, yet in order to remain a minimum rank of 2, $b$ must equal 0 and $a$ and $d$ can be any positive real numbers.

### 4-2.1 $S^Z_p$ (Sign Pattern Matrices in $Z_p$)

We now consider $2 \times 2$ matrices over the finite field $Z_p$ where $p$ is a prime number. How many sign pattern matrices are there in $Z_p$? What is the minimum rank of a sign pattern in $Z_p$?

The finite field $Z_p$ presents some unique properties that are worth mentioning. First, there are no negative numbers that can be allowed. Yet, negative numbers can be the result of a calculation to determine the value of the determinant. In the case where the determinant calculated is negative the properties of $\text{mod}$ are applied and the corresponding nonnegative element in $Z_p$ is used as the value of the determinant. See the example below.

Example: Let $S^Z_5 = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$. In this case $ad$ is 0, and $bc$ can be any combination of $\{1, 2, 3, 4\}$. When $b = 4$ and $c = 3$, 

\[ \det(S^{\mathbb{Z}}) = -12 \equiv 3. \]

Note that the negative numbers will result in an equivalent element that is in the field \( \mathbb{Z}_5 \) and will never be equal to zero. This is true for all \( \mathbb{Z}_p \) where \( p \) is a prime number.

Note that for all of the following sign patterns the zero matrix \( S_0^{\mathbb{Z}_p} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) \( mr(S_0^{\mathbb{Z}_p}) = 0 \) is excluded to save space and time.

Consider \( S^{\mathbb{Z}_2} \) where possibilities for each entry is in \( \{0, +\} \). There are a total of 15 possible sign pattern matrices.

1. \( mr(S^{\mathbb{Z}_2}) = 1 : \)
   
   a. \( S_1^{\mathbb{Z}_2} = \begin{bmatrix} 0 & 0 \\ 0 & + \end{bmatrix} \) or \( \begin{bmatrix} 0 & + \\ 0 & 0 \end{bmatrix} \) or \( \begin{bmatrix} + & 0 \\ 0 & 0 \end{bmatrix} \) or \( \begin{bmatrix} 0 & 0 \\ + & 0 \end{bmatrix} \) where any 3 positions are 0 and \( |S_1^{\mathbb{Z}_2}| = 4. \)

   b. \( S_2^{\mathbb{Z}_2} = \begin{bmatrix} 0 & 0 \\ + & + \end{bmatrix} \) or \( \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix} \) or \( \begin{bmatrix} + & + \\ 0 & 0 \end{bmatrix} \) or \( \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \) where a column or a row is zero and \( |S_2^{\mathbb{Z}_2}| = 4. \)

   c. \( S_3^{\mathbb{Z}_2} = \begin{bmatrix} + & + \\ + & + \end{bmatrix} \) where all entries are + and \( |S_3^{\mathbb{Z}_2}| = 1. \)

2. \( mr(S^{\mathbb{Z}_2}) = 2 : \)
   
   a. \( S_4^{\mathbb{Z}_2} = \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix} \) where 2 zeros along the diagonal (similar to the example above) and \( |S_4^{\mathbb{Z}_2}| = 1. \)

   b. \( S_5^{\mathbb{Z}_2} = \begin{bmatrix} + & + \\ 0 & + \end{bmatrix} \) where any one entry is 0 and \( |S_5^{\mathbb{Z}_2}| = 4. \)

   c. \( S_6^{\mathbb{Z}_2} = \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \) where 2 off-diagonal entries are zeros, \( b, c = 0 \) and \( |S_6^{\mathbb{Z}_2}| = 1. \)

4-2.2 \( S_{sym}^{\mathbb{Z}_p} \) (Symmetric Sign Pattern Matrices in \( \mathbb{Z}_p \))

We now analyze the symmetric case in \( \mathbb{Z}_p \) field. Let \( S_{sym}^{\mathbb{Z}_p} = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \). The symmetric sign pattern matrices in the finite field \( \mathbb{Z}_p \) narrows the number of possibilities to 5.

1. \( mr(S_{sym}^{\mathbb{Z}_2}) = 1 : \)
   
   a. \( S_{sym-1}^{\mathbb{Z}_2} = \begin{bmatrix} 0 & 0 \\ 0 & + \end{bmatrix} \) or \( \begin{bmatrix} + & 0 \\ 0 & 0 \end{bmatrix} \) and \( |S_{sym-1}^{\mathbb{Z}_2}| = 2. \)

   b. \( S_{sym-2}^{\mathbb{Z}_2} = \begin{bmatrix} + & + \\ + & + \end{bmatrix} \) and \( |S_{sym-2}^{\mathbb{Z}_2}| = 1. \)

2. \( mr(S_{sym}^{\mathbb{Z}_2}) = 2 : \)
\[ \begin{align*}
a. \quad S_{\text{sym-3}}^{\mathbb{Z}_p} &= \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix} \text{ and } |S_{\text{sym-3}}^{\mathbb{Z}_p}| = 1. \\
b. \quad S_{\text{sym-4}}^{\mathbb{Z}_p} &= \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \text{ and } |S_{\text{sym-4}}^{\mathbb{Z}_p}| = 1.
\end{align*} \]

### 4-2.3 \( S_{+\text{def}}^{\mathbb{Z}_p} \) (Positive Definite Sign Pattern Matrices in \( \mathbb{Z}_p \))

Again similar to that of the field \( \mathbb{R} \), of the above sign pattern matrices those with \( mr(S_{\text{sym}}^{\mathbb{Z}_p}) = 2 \) (total 2) are by definition the only sign patterns that are eligible to be positive definite due to the fact that all \( 2 \times 2 \) positive definite matrices are rank 2. The symmetric positive definite sign pattern matrices in \( \mathbb{Z}_p \) field has similar conditions as those stated in the real in section 4-1.3. However, the finite field \( \mathbb{Z}_p \) again brings some new restrictions as to the calculation of some items. Below are the new conditions in \( \mathbb{Z}_p \).

Since both \( \lambda_1 \) and \( \lambda_2 \) are positive and they cannot be equal to \( p \) as it would result in a zero, we have the following conditions:

(iii) \( \lambda_1 + \lambda_2 = [(a + d) \mod p] > 0 \); and

(iv) \( \lambda_1 \lambda_2 = [(ad - b^2) \mod p] > 0 \).

The following matrices are symmetric and of rank 2, yet neither of them meet the above conditions for positive definite. Condition (iii) fails in both the cases below.

1. **a.** \( S_{\text{sym-3}}^{\mathbb{Z}_p} = \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix} \). In this case \( a + d \) equals zero, thus failing condition (iii)

2. **b.** \( S_{\text{sym-4}}^{\mathbb{Z}_p} = \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \). In this case again, \( a + d > 0 \) fails. The reason is as follows: because \( (a + (p - a)) \mod p = 0, a + (p - a) \) will always be mapped to \( p \).

The situation in b. is quite unique and is a result of the field \( \mathbb{Z}_p \).

When we consider the positive semi-definite matrices, conditions (iii) and (iv) become:

(v) \( \lambda_1 + \lambda_2 = [(a + d) \mod p] \geq 0 \); and

(vi) \( \lambda_1 \lambda_2 = [(ad - b^2) \mod p] \geq 0 \).

Both \( S_{\text{sym-4}}^{\mathbb{Z}_p} = \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \) and \( S_{\text{sym-3}}^{\mathbb{Z}_p} = \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix} \) satisfy conditions (v) and (vi). Therefore, \( S_{\text{sym-4}}^{\mathbb{Z}_p} \) and \( S_{\text{sym-3}}^{\mathbb{Z}_p} \) are **symmetric positive semi-definite sign pattern matrices** in the field \( \mathbb{Z}_p \).

### 5. Conclusion

The minimum rank of a \( 2 \times 2 \) sign pattern matrix in \( \mathbb{R} \) allowed for 81 different possible sign patterns that when sorted by their minimum rank yielded 1 of minimum rank 0, 32 of minimum rank 1, and 48 of minimum rank 2. Further isolating by symmetry allowed for a unique restriction to symmetric positive definite, which required the minimum rank to be that of 2. The number of symmetric sign pattern matrices in the real was 27, of the 27 only 18 were of minimum rank 2. The 18 symmetric sign pattern matrices with minimum rank of 2 were narrowed to those sign pattern matrices with positive eigenvalues in both product and sum. In the real there was only 1 sign pattern that fit that of symmetric positive definite, \( S_{+\text{def}}^{\mathbb{R}} = \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \). The minimum rank was then applied to the
field \( \mathbb{Z}_p \). Beginning with only 16 possible 2x2 sign patterns in \( \mathbb{Z}_p \) based on minimum rank the number of sign pattern matrices can be reduced to 5 that are symmetric. The symmetric sign pattern matrices that were of minimum rank 2 were then screened for positive definite. Surprisingly the 2 sign pattern matrices did not meet the requirements of a positive definite sign pattern matrix in the field \( \mathbb{Z}_p \). The discovery that no 2x2 sign pattern exists in the positive definite finite field is one that is quite surprising. Allowing the restriction to positive semi-definite there were 2 symmetric matrices that qualify as semi-definite sign pattern matrices. The minimum rank of the sign pattern matrices is an important tool in isolating sign patterns in order to find a specific pattern of signs.

The minimum rank used here to determine the sign pattern matrix that was positive definite can also be used, as mentioned in the introduction, to find the sign patterns without actual numbers that are stable or that give an indication of a certain property in 2 or more dimensions. The usefulness of this reduction is seen in fields like the stock market to find the stable pattern in the market and predict future outcomes. The minimum rank of a sign pattern matrix is important to the math world and also to others.

**Works Cited**


